

A new view on boundary conditions in the Grioli-Koiter-Mindlin-Toupin indeterminate couple stress model

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Abstract

In this paper we consider the Grioli-Koiter-Mindlin-Toupin linear isotropic indeterminate couple stress model. Our main aim is to show that, up to now, the boundary conditions have not been completely understood for this model. As it turns out, and to our own surprise, restricting the well known boundary conditions stemming from the strain gradient or second gradient models to the particular case of the indeterminate couple stress model, does not always reduce to the Grioli-Koiter-Mindlin-Toupin set of accepted boundary conditions. We present, therefore, a proof of the fact that when specific “mixed” kinematical and traction boundary conditions are assigned on the boundary, no “a priori” equivalence can be established between Mindlin’s and our approach.

Key words: generalized continua, second gradient elasticity, strain gradient elasticity, couple stresses, hyperstresses, indeterminate couple stress model, consistency of mixed boundary conditions.

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Contents

1	Introduction	3
1.1	Notational agreements	4
1.2	The indeterminate couple stress model	5
1.3	Brief digression concerning differential geometry	7
1.4	Weakly and strongly independent surface fields	8
2	Bulk equations and (Mindlin’s weakly independent) boundary conditions in the indeterminate couple stress model	9
2.1	Equilibrium and constitutive equations of the indeterminate couple stress model	9
2.2	Classical (weakly independent) boundary conditions in the indeterminate couple stress model . .	11
3	Through second gradient elasticity towards the indeterminate couple stress theory: a direct approach	14
3.1	Second gradient model: general variational setting	14
3.2	The indeterminate couple stress model viewed as a subclass of the second gradient elasticity model	16
3.3	Reduction from the third order hyperstress tensor $\tilde{\mathbf{m}}$ to Mindlin’s second order couple stress tensor $\tilde{\mathbf{m}}$	18
3.4	A direct way to obtain strongly independent boundary conditions in the indeterminate couple model	19
3.5	The geometric and traction, strongly independent, boundary conditions for the indeterminate couple stress model	22
4	Assessment of the strongly independent boundary conditions for the indeterminate couple stress model in a form directly comparable to Mindlin and Tiersten’s ones	23
4.1	Towards a direct comparison with Mindlin’s traction boundary conditions	23
4.2	Final form of the strongly independent, geometric and traction boundary conditions for the indeterminate couple stress model	23
5	Are Mindlin and Tiersten’s weakly independent boundary conditions equivalent to our strongly independent ones?	24
5.1	Fully kinematical boundary conditions	26
5.2	Fully traction boundary conditions	26
5.3	Mixed 1: displacement/double-force boundary conditions	27
5.4	Mixed 2: force/ $D^1(u)$ boundary conditions	27
6	Conclusions	27
	References	28
	Appendix	31
A.1	First variation of a second gradient action functional and principle of virtual power in compact form	31
A.2	Some useful relationships between the third order hyperstress tensor and the second order couple stress tensor for the indeterminate couple stress model	32
A.3	Some alternative calculations useful to rewrite the governing equations and boundary conditions in a form which is directly comparable to Mindlin’s one	33
A.4	The missing steps in Mindlin and Tiersten’s classical approach	34
A.5	The proof of Lemma 2.3	36
A.6	The proof of some identities to ulteriorly simplify the traction boundary conditions	37
A.7	The proof of Proposition 4.1	38
A.8	Some lemmas useful to understand Mindlin and Tiersten’s approach	39
A.9	Concluding diagrams	40

1 Introduction

Higher gradient elasticity models are nowadays increasingly used to describe mechanical systems with underlying micro- or nano-structures (see e.g. among many others [7, 80, 4, 13, 16, 24, 5, 23, 78, 76, 20, 29, 28, 48, 3]) or to regularize certain ill-posed problems with higher gradient contributions (see e.g. [58, 63, 22, 21, 31, 65]). Such higher gradient models, together with the more general class of micromorphic models [67, 62, 64, 70, 68, 46, 84, 8], have been also proved to be a useful tool for the description of micro-structured materials showing exotic behaviours in the dynamic regime (see e.g. [54, 53, 14, 77, 79, 10, 34, 33, 89]).

One among such higher order models which was introduced at the very beginning is the so called *indeterminate couple stress model* in which the higher gradient contributions only enter through gradients on the continuum rotation. We place ourselves in the context of the linear elastic, isotropic model by choosing a specific form of the quadratic free elastic energy density.

The question of boundary conditions in higher gradient elasticity models has been a subject of constant attention. The matter is that in a higher gradient model, it is not possible to independently vary the test function and its gradient. Some sort of split into tangential and normal parts is usually performed (see e.g. [17, 18, 19]). This is well known in general higher gradient models. The boundary conditions in the general case of gradient elasticity and strain gradient elasticity have been settled in the paper by Bleustein [9], see also [57, 52, 51, 12]. However, as it turns out, the boundary conditions obtained by Tiersten and Bleustein in [85] with respect to the special case of the indeterminate couple stress model are not the only possible ones in the framework of the indeterminate couple-stress model.

While the strain gradient framework necessitates to work with a third order hyperstress tensor, the indeterminate couple stress model is apparently simpler: it restricts the form of the curvature energy and allows to work with a second order couple-stress tensor work-conjugate to gradients of rotation. For this apparent simplification the indeterminate couple stress model has been heavily investigated and is still being heavily used as well. A first answer as regards boundary conditions has been given by Mindlin and Tiersten as well as Koiter [59, 47] who established (correctly) that only 5 geometric and 5 traction boundary conditions can be prescribed. Their format of boundary conditions has become the commonly accepted one for the couple stress model [59, 86, 47, 6, 88, 74], all these papers using the same set of (incomplete) boundary conditions. It seems, to us, however, that the state of the art in general strain gradient theories [59, 86, 47, 72, 75, 6, 56, 55, 44, 64] is much more advanced as far as boundary conditions are concerned.

This paper has been motivated by our reading of [42, 40, 41, 39, 38, 43], in which the form of traction boundary conditions in the indeterminate couple stress model, together with an apparently plausible physical postulate lead to unacceptable conclusions, see [73]. Therefore, there had to be an underlying problem which we believe to have tracked down to the hitherto accepted format of boundary conditions.

The main result of this paper, consisting in setting up a "strongly independent" set of boundary conditions for the couple stress model, has been announced in [66]. This contribution is now structured as follows: after a subsection fixing the notations used throughout the paper, we outline some related models in isotropic second gradient elasticity and we give a brief digression concerning differential geometry. In Section 2, we present the equilibrium equations and the constitutive equations of the indeterminate couple stress model as they have been derived in the literature. We also present the classical "weakly independent" boundary conditions proposed by Mindlin and Tiersten [59] and the main arguments of their proposal. Since we remark that these boundary conditions are not the only possible ones, in Section 3 we obtain the novel set of boundary conditions in the indeterminate couple stress model. To this main aim of our paper, we follow two different paths. On the one hand, we consider the indeterminate couple stress model as a special case of the second gradient elasticity model and we derive the "strongly independent" boundary conditions which follow naturally as restriction of such general framework, see Subsection 3.2. This kind of approach involves the third order hyperstress tensor as a reminiscence of the second gradient elasticity approach. However, in Subsection 3.3, we prove that the equilibrium equations and the boundary conditions may all be rewritten in terms of Mindlin's second order couple stress tensor. On the other hand, following the line of Mindlin's argument in combination with some calculations specific to second gradient elasticity model, we set up a "direct approach" which leads to the same set of boundary conditions with those coming from second gradient elasticity, see Section 3.4. However, these boundary conditions do not always coincide with those proposed by Mindlin and Tiersten and which are accepted and used until now in the literature. We explain this fact in Subsection 1.4 where we explicitly show that, if an "a priori" equivalence can be found in most cases between our approach and Mindlin's one, this is

not the case when considering "mixed" boundary conditions, simultaneously assigning the force and the curl of displacement (Mindlin) or the force and the normal derivative of displacement (our approach) on the same portion of the boundary.

In the appendix, we give some explicit or alternative calculations which are used in the main text and we answer again to the question: what are the missing steps in Mindlin and Tiersten's approach? (see Appendix A.4). We do so by using all the arguments provided throughout the paper and pointed out in different circumstances. We end our paper by some concluding diagrams summarizing our findings.

1.1 Notational agreements

In this paper, we denote by $\mathbb{R}^{3 \times 3}$ the set of real 3×3 second order tensors, which will be written with capital letters. We denote respectively by \cdot , $:$ and $\langle \cdot, \cdot \rangle$ a simple and double contraction and the scalar product between two tensors of any suitable order¹. Everywhere we adopt the Einstein convention of sum over repeated indices if not differently specified. The standard Euclidean scalar product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(X \cdot Y^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3 \times 3}$. The identity tensor on $\mathbb{R}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$. We adopt the usual abbreviations of Lie-algebra theory, i.e., $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid X^T = -X\}$ is the Lie-algebra of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \text{tr}(X) = 0\}$ is the Lie-algebra of traceless tensors. For all $X \in \mathbb{R}^{3 \times 3}$ we set $\text{sym } X = \frac{1}{2}(X^T + X) \in \text{Sym}$, $\text{skew } X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$ and the deviatoric part $\text{dev } X = X - \frac{1}{3} \text{tr}(X) \mathbb{1} \in \mathfrak{sl}(3)$ and we have the *orthogonal Cartan-decomposition of the Lie-algebra* $\mathfrak{gl}(3)$

$$\mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \text{Sym}(3)\} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbb{1}, \quad X = \text{dev sym } X + \text{skew } X + \frac{1}{3} \text{tr}(X) \mathbb{1}. \quad (1.1)$$

Throughout this paper (when we do not specify else) Latin subscripts take the values 1, 2, 3. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate. Here, for

$$\bar{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \quad (1.2)$$

we consider the operators $\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ and $\text{anti} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ which verify the following identities

$$\begin{aligned} \text{axl}(\bar{A}) &:= (a_1, a_2, a_3)^T, & \bar{A} \cdot v &= (\text{axl } \bar{A}) \times v, & (\text{anti}(v))_{ij} &= -\epsilon_{ijk} v_k, & \forall v \in \mathbb{R}^3, \\ (\text{axl } \bar{A})_k &= -\frac{1}{2} \epsilon_{ijk} \bar{A}_{ij} = \frac{1}{2} \epsilon_{kij} \bar{A}_{ji}, & \bar{A}_{ij} &= -\epsilon_{ijk} (\text{axl } \bar{A})_k =: \text{anti}(\text{axl } \bar{A})_{ij}, & (a \times b)_i &= \epsilon_{ijk} a_j b_k, \end{aligned} \quad (1.3)$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita third order permutation tensor.

We consider a body which occupies a bounded open set Ω of the three-dimensional Euclidian space \mathbb{R}^3 and assume that its boundary $\partial\Omega$ is a smooth surface of class C^2 . An elastic material fills the domain $\Omega \subset \mathbb{R}^3$ and we refer the motion of the body to rectangular axes Ox_i .

With reference to Fig.1, n is the outward unit normal to $\partial\Omega$, Γ is an open subset of the boundary $\partial\Omega$, ν^- is a vector tangential to the surface $\partial\Omega \setminus \bar{\Gamma}$ and which is orthogonal to its boundary $\partial(\partial\Omega \setminus \bar{\Gamma})$, $\tau^- = n \times \nu^-$ is the tangent to the curve $\partial(\partial\Omega \setminus \bar{\Gamma})$ with respect to the orientation on $\partial\Omega \setminus \bar{\Gamma}$ given by the outward unit normal n to this surface. Similarly, ν^+ is a vector tangential to the surface Γ and which is orthogonal to its boundary $\partial\Gamma$, $\tau^+ = n \times \nu^+$ is the tangent to the curve $\partial\Gamma$ with respect to the orientation on Γ .

In the following, given any vector field a defined on the boundary $\partial\Omega$ we will also set

$$\llbracket \langle a, \nu \rangle \rrbracket := \langle a^+, \nu^+ \rangle + \langle a^-, \nu^- \rangle = \langle a^+, \nu \rangle - \langle a^-, \nu \rangle = \langle a^+ - a^-, \nu \rangle, \quad (1.4)$$

which defines a measure of the jump of a through the line $\partial\Gamma$, where $\nu := \nu^+ = -\nu^-$ and

$$[\cdot]^- := \lim_{\substack{x \in \partial\Omega \setminus \bar{\Gamma} \\ x \rightarrow \partial\Gamma}} [\cdot], \quad [\cdot]^+ := \lim_{\substack{x \in \Gamma \\ x \rightarrow \partial\Gamma}} [\cdot].$$

¹For example, $(A \cdot v)_i = A_{ij} v_j$, $(A \cdot B)_{ik} = A_{ij} B_{jk}$, $A : B = A_{ij} B_{ji}$, $(C \cdot B)_{ijk} = C_{ijp} B_{pk}$, $(C : B)_i = C_{ijp} B_{pj}$, $\langle v, w \rangle = v \cdot w = v_i w_i$, $\langle A, B \rangle = A_{ij} B_{ij}$ etc.

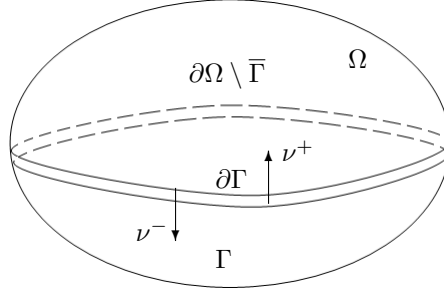


Figure 1: The domain $\Omega \subset \mathbb{R}^3$ together with the part $\Gamma \subset \partial\Omega$, where Dirichlet boundary conditions are prescribed. We need to represent the boundary conditions on a disjoint union of $\partial\Omega = (\partial\Omega \setminus \bar{\Gamma}) \cup \Gamma \cup \partial\Gamma$, where Γ is a open subset of $\partial\Omega$.

We are assuming here that $\partial\Omega$ is a smooth surface. Hence, there are no geometric singularities of the boundary. The jump $[[\cdot]]$ arises only as a consequence of possible discontinuities which follows from the prescribed boundary conditions on Γ and $\partial\Omega \setminus \bar{\Gamma}$. Nevertheless, if one would like to explicitly consider continua with non-smooth boundaries, the jump conditions to be imposed at the edges of the boundary would be formally the same to those that we will present in the remainder of this paper, with the precision that the jump would in this case indicate a true jump across a geometrical discontinuity of the surface.

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in \mathbb{R} , \mathbb{R}^3 or $\mathbb{R}^{3 \times 3}$, respectively will be denoted by $L^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces [1, 32, 49]

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega) \mid \text{grad } u \in L^2(\Omega)\}, & \|u\|_{H^1(\Omega)}^2 &:= \|u\|_{L^2(\Omega)}^2 + \|\text{grad } u\|_{L^2(\Omega)}^2, \\ H(\text{curl}; \Omega) &= \{v \in L^2(\Omega) \mid \text{curl } v \in L^2(\Omega)\}, & \|v\|_{H(\text{curl}; \Omega)}^2 &:= \|v\|_{L^2(\Omega)}^2 + \|\text{curl } v\|_{L^2(\Omega)}^2, \\ H(\text{div}; \Omega) &= \{v \in L^2(\Omega) \mid \text{div } v \in L^2(\Omega)\}, & \|v\|_{H(\text{div}; \Omega)}^2 &:= \|v\|_{L^2(\Omega)}^2 + \|\text{div } v\|_{L^2(\Omega)}^2, \end{aligned} \quad (1.5)$$

of functions u or vector fields v , respectively.

For vector fields v with components in $H^1(\Omega)$, i.e. $v = (v_1, v_2, v_3)^T$, $v_i \in H^1(\Omega)$, we define $\nabla v = ((\nabla v_1)^T, (\nabla v_2)^T, (\nabla v_3)^T)^T$, while for tensor fields P with rows in $H(\text{curl}; \Omega)$, resp. $H(\text{div}; \Omega)$, i.e. $P = (P_1^T, P_2^T, P_3^T)$, $P_i \in H(\text{curl}; \Omega)$ resp. $P_i \in H(\text{div}; \Omega)$ we define $\text{Curl } P = ((\text{curl } P_1)^T, (\text{curl } P_2)^T, (\text{curl } P_3)^T)^T$, $\text{Div } P = (\text{div } P_1, \text{div } P_2, \text{div } P_3)^T$. The corresponding Sobolev-spaces will be denoted by

$$H^1(\Omega), \quad H^1(\text{Div}; \Omega), \quad H^1(\text{Curl}; \Omega).$$

1.2 The indeterminate couple stress model

In the indeterminate couple stress model we consider that the elastic energy is given in the form

$$W = W_{\text{lin}}(\nabla u) + W_{\text{curv}}(\nabla[\text{axl}(\text{skew } \nabla u)]) = W_{\text{lin}}(\nabla u) + \widetilde{W}_{\text{curv}}(\nabla \text{curl } u), \quad (1.6)$$

where

$$W_{\text{lin}}(\nabla u) = \mu \|\text{sym } \nabla u\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym } \nabla u)]^2 = \mu \|\text{dev sym } \nabla u\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym } \nabla u)]^2. \quad (1.7)$$

Here, $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the infinitesimal bulk modulus with λ the first Lamé constant. In order to discuss the form of the curvature energy $W_{\text{curv}}(\nabla[\text{axl}(\text{skew } \nabla u)])$, let us recall some variants of the linear isotropic indeterminate couple stress models. Some parts of this classification have already been included in the paper [30] but we include them also here for the sake of completeness:

- **the indeterminate couple stress model** (Grioli-Koiter-Mindlin-Toupin model) [35, 2, 47, 59, 87, 82, 36] in which the higher derivatives (apparently) appear only through derivatives of the infinitesimal continuum

rotation $\text{curl } u$. Hence, the curvature energy has the equivalent forms

$$\begin{aligned} W_{\text{curv}}(\nabla[\text{axl}(\text{skew } \nabla u)]) &= \frac{\alpha_1}{4} \|\text{sym } \nabla \text{curl } u\|^2 + \frac{\alpha_2}{4} \|\text{skew } \nabla \text{curl } u\|^2 \\ &= \alpha_1 \|\text{sym } \nabla[\text{axl}(\text{skew } \nabla u)]\|^2 + \alpha_2 \|\text{skew } \nabla[\text{axl}(\text{skew } \nabla u)]\|^2 \\ &= \frac{\alpha_1}{4} \|\text{dev sym } \nabla \text{curl } u\|^2 + \frac{\alpha_2}{4} \|\text{skew } \nabla \text{curl } u\|^2. \end{aligned} \quad (1.8)$$

Here, we have used the identities

$$2 \text{axl}(\text{skew } \nabla u) = \text{curl } u, \quad \text{tr}[\nabla[\text{axl}(\text{skew } \nabla u)]] = \frac{1}{2} \text{tr}[\nabla[\text{curl } u]] = \frac{1}{2} \text{div}[\text{curl } u] = 0, \quad (1.9)$$

together with the fact that $\nabla \text{curl } u$ is a trace-free second order tensor and hence so is $\text{sym } \nabla \text{curl } u$. This implies that $\text{dev sym } \nabla \text{curl } u = \text{sym } \nabla \text{curl } u$. Although this energy admits the equivalent forms $(1.8)_1$ and $(1.8)_3$, the equations and the boundary value problem of the indeterminate couple stress model is usually formulated only using the form $(1.8)_1$ of the energy. Hence, we may individuate one of the aims of the present paper in the fact that we want to formulate the boundary value problem for **the indeterminate couple stress model using the alternative form $(1.8)_2$ of the energy** of the Grioli-Koiter-Mindlin-Toupin model (see Section 2). We also remark that the spherical part of the couple stress tensor is zero since $\text{tr}(\nabla \text{curl } u) = \text{div}(\text{curl } u) = 0$. In order to prove the pointwise uniform positive definiteness it is assumed, following [47], that $\alpha_1 > 0, \alpha_2 > 0$ (corresponds to $-1 < \eta := \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} < 1$ in the notation of [47]). Note that pointwise uniform positivity is often assumed [47] when deriving analytical solutions for simple boundary value problems because it allows to invert the couple stress-curvature relation. We have shown elsewhere [30] that pointwise positive definiteness is not necessary for well-posedness.

- In this setting, **Grioli** [35, 36] (see also Fleck [25, 26, 27]) initially considered only the choice $\alpha_1 = \alpha_2$. In fact, the energy originally proposed by Grioli [35] is

$$\begin{aligned} W_{\text{curv}}(D^2 u) &= \mu L_c^2 \frac{\alpha_1}{4} [\|\nabla(\text{curl } u)\|^2 + \eta \text{tr}[(\nabla(\text{curl } u))^2]] \\ &= \mu L_c^2 \frac{\alpha_1}{4} [\|\text{dev sym } \nabla[\text{axl}(\text{skew } \nabla u)]\|^2 + \|\text{skew } \nabla[\text{axl}(\text{skew } \nabla u)]\|^2 \\ &\quad + \eta \langle \nabla[\text{axl}(\text{skew } \nabla u)], (\nabla[\text{axl}(\text{skew } \nabla u)))^T \rangle] \\ &= \mu L_c^2 \frac{\alpha_1}{4} [\|\text{dev sym } \nabla(\text{curl } u)\|^2 + \|\text{skew } \nabla(\text{curl } u)\|^2 \\ &\quad + \eta \langle \nabla(\text{curl } u), (\nabla(\text{curl } u))^T \rangle] \\ &= \mu L_c^2 \frac{\alpha_1}{4} [(1 + \eta) \|\text{dev sym } \nabla(\text{curl } u)\|^2 + (1 - \eta) \|\text{skew } \nabla(\text{curl } u)\|^2]. \end{aligned} \quad (1.10)$$

Mindlin [59, p. 425] (with $\eta = 0$) explained the relations between Toupin's constitutive equations [86] and Grioli's [35] constitutive equations and concluded that the obtained equations in the linearized theory are identical, since the extra constitutive parameter η of Grioli's model does not explicitly appear in the equations of motion but enters only the boundary conditions, since $\nabla \text{axl}(\text{skew } \nabla u) = [\text{Curl}(\text{sym } \nabla u)]^T$, $\text{Div Curl}(\cdot) = 0$, and

$$\text{Div}\{\text{anti Div}[\nabla \text{axl}(\text{skew } \nabla u)]^T\} = \text{Div}\{\text{anti Div}[\text{Curl}(\text{sym } \nabla u)]\} = \text{Div}\{\text{anti}(0)\} = 0.$$

The same extra constitutive coefficient appears in Mindlin and Eshel [60] and Grioli's version (1.10).

- **the modified - symmetric couple stress model - the conformal model.** On the other hand, in the conformal case [72, 71] one may consider that $\alpha_2 = 0$, which makes the second order couple stress tensor \tilde{m} symmetric and trace free [11]. This conformal curvature case has been considered by Neff in [72], the curvature energy having the form

$$\widetilde{W}_{\text{curv}}(\nabla \text{curl } u) = \frac{\alpha_1}{4} \|\text{sym } \nabla \text{curl } u\|^2. \quad (1.11)$$

Indeed, there are two major reasons uncovered in [72] for using the modified couple stress model. First, in order to avoid singular stiffening behaviour for smaller and smaller samples in bending [69] one has to take $\alpha_2 = 0$. Second, based on a homogenization procedure invoking an intuitively appealing natural “micro-randomness” assumption (a strong statement of microstructural isotropy) requires conformal invariance, which is again equivalent to $\alpha_2 = 0$. Such a model is still well-posed [45] leading to existence and uniqueness results with only one additional material length scale parameter, while it is **not** pointwise uniformly positive definite. The initial motivation of Yang et al. [88] for using the modified couple stress model is based on incomplete arguments [61], even if their conclusions concerning a symmetric couple stress tensor may be kept in some particular phenomenological cases [61].

1.3 Brief digression concerning differential geometry

When dealing with higher order theories it is suitable to introduce (see also [81, 15] for details) two second order tensors T and Q which are the two projectors on the tangent plane and on the normal to the considered surface, respectively. As it is well known from differential geometry, such projectors actually allow to split a given vector or tensor field in one part projected on the plane tangent to the considered surface and one projected on the normal to such surface. We can introduce the quoted projectors as

$$T = \tau \otimes \tau + \nu \otimes \nu = \mathbb{1} - n \otimes n, \quad Q = n \otimes n.$$

It is easy to check that the following identities are verified by the two introduced projectors

$$T + Q = \mathbb{1}, \quad T \cdot T = T, \quad Q \cdot Q = Q, \quad T \cdot Q = 0, \quad T = T^T, \quad Q = Q^T. \quad (1.12)$$

It is then straightforward that any vector v can be represented in the local basis $\{\tau, n, \nu\}$ as

$$v = v \cdot (T + Q) = (T + Q) \cdot v = \underbrace{\langle v, \tau \rangle \tau + \langle v, \nu \rangle \nu}_{T \cdot v} + \underbrace{\langle v, n \rangle n}_{Q \cdot v},$$

or equivalently the components of v can be written as

$$v_i = v_h T_{hi} + v_h Q_{hi} = T_{ih} v_h + Q_{ih} v_h.$$

Analogously, a second order tensor B can be represented in such local coordinates as $B = (T + Q)^T \cdot B \cdot (T + Q)$ and this representation can be also straightforwardly generalized to a tensor of generic order N . To the sake of generality, from now on we introduce a global orthonormal basis $\{e^1, e^2, e^3\}$ in which all fields (included τ, ν and n) will be represented if not differently specified. This also implies that all the space differential operations which will be performed in the following calculations are all referred to the coordinates (X_1, X_2, X_3) associated to such global basis.

From [37, p. 58, ex. 7], we have the following variant of the surface divergence theorem

Proposition 1.1. *Suppose that w satisfies $\langle w, n \rangle = 0$ on the surface $S \subset \mathbb{R}^3$, then*

$$\int_S \langle T, \nabla w \rangle da = \int_{\partial S} \langle n \times w, \tau \rangle ds, \quad (1.13)$$

where τ is a vector tangent to S and to ∂S .

Taking now $w = T \cdot v$, for arbitrary $v \in \mathbb{R}^3$ and using that

$$\langle n \times (T \cdot v), \tau \rangle = \langle \tau \times n, T \cdot v \rangle =: \langle \nu, T \cdot v \rangle = \langle T \cdot \nu, v \rangle$$

we obtain

$$\int_S \langle T, \nabla (T \cdot v) \rangle da = \int_{\partial S} \langle n \times (T \cdot v), \tau \rangle ds = \int_{\partial S} \langle T \cdot \nu, v \rangle ds = \int_{\partial S} \langle \nu, v \rangle ds, \quad (1.14)$$

for $v \in \mathbb{R}^3$, since $T \cdot \nu = \nu \cdot T = \nu$. We explicitly remark that the vector ν is tangent to the surface S but orthogonal to its boundary ∂S (it points inward or outwards, depending on the choice of the orientation of n

and τ : usually, n is chosen to be the outward unit normal to S and the orientation of τ is chosen in order to have that the vector ν , orthogonal to the border of S , is pointing outward the border of the surface S itself). With such notations, when considering a vector field v defined in the vicinity of the considered surface, the surface divergence theorem can be applied to the projection of v on the tangent plane to the surface as follows (see e.g. [83])

$$\int_S \text{Div}^S(v) da := \int_S \langle T, \nabla(T \cdot v) \rangle da = \int_{\partial S} \langle T \cdot v, \nu \rangle da = \int_{\partial S} \langle v, T \cdot \nu \rangle da = \int_{\partial S} \langle v, \nu \rangle ds, \quad (1.15)$$

where, clearly, in the last equality the notation

$$\text{Div}^S(v) := \langle \nabla(T \cdot v), T \rangle$$

for the surface divergence of a generic vector v has been used. Equivalently, in index notation the surface divergence theorem reads

$$\int_S (T_{ij} v_j)_{,k} T_{ik} da = \int_{\partial S} v_i T_{ik} \nu_k ds = \int_{\partial S} v_i \nu_i ds. \quad (1.16)$$

This definition of surface divergence can be extended to higher order tensors, in particular, for a second order tensor B , its surface divergence is introduced as the vector of components $[\text{Div}^S(B)]_i = (T \cdot B)_{ij,k} T_{jk}$.

Remark 1.2. *We explicitly remark that if S coincides with the boundary $\partial\Omega$ of the considered body and Γ is an open subset of $\partial\Omega$, then the surface divergence theorem (1.15) implies (see Fig. 1 and eq. (1.4))*

$$\begin{aligned} \int_{\partial\Omega} \text{Div}^S(v) da &:= \int_{\partial\Omega} \langle T, \nabla(T \cdot v) \rangle da = \int_{\partial\Omega \setminus \Gamma} \text{Div}^S(v^-) da + \int_{\Gamma} \text{Div}^S(v^+) da \\ &= \int_{\partial(\partial\Omega \setminus \Gamma)} \langle v^-, \nu^- \rangle ds + \int_{\partial\Gamma} \langle v^+, \nu^+ \rangle ds = \int_{\partial\Gamma} \llbracket \langle v, \nu \rangle \rrbracket ds. \end{aligned} \quad (1.17)$$

Clearly, we can equivalently write in index notation

$$\int_{\partial\Omega} (T_{ij} v_j)_{,k} T_{ik} da = \int_{\partial\Gamma} \llbracket v_j \nu_j \rrbracket ds. \quad (1.18)$$

1.4 Weakly and strongly independent surface fields

Since it will be useful in the following, we give in this subsection some definitions which will be used throughout the paper. In particular, we introduce the notions of "strongly" and "weakly independent" vector fields defined on suitable regular surfaces.

Definition 1.3. [Weakly independent surface fields] *Given two vector fields u and v defined on a suitable regular surface S , we say that they are "weakly independent" if we can arbitrarily assign u and v , independent of each other, by choosing two vectors \bar{u} and \bar{v} and a function f such that*

$$u = \bar{u} \quad \text{and} \quad v = f(\bar{u}, \bar{v}). \quad (1.19)$$

By means of the notion of weak independence, we can arbitrarily fix the vectors u and v on the surface S , but a variation of the chosen \bar{u} induces a variation on one part of v . Nevertheless, the vector v can still be arbitrarily chosen thanks to the arbitrariness of \bar{v} and f .

Definition 1.4. [Strongly independent surface fields] *Given two vector fields u and v defined on a suitable regular surface S , we say that they are "strongly independent" if we can arbitrarily assign u and v , independent of each other, by choosing two vectors \bar{u} and \bar{v} such that*

$$u = \bar{u} \quad \text{and} \quad v = \bar{v}. \quad (1.20)$$

The notion of strong independence allows to arbitrarily fix the two vector fields u and v on the surface S and a variation in the choice of the first vector $u = \bar{u}$ does not induce a variation in the vector v .

Considering the power of external actions for a second gradient continuum

$$\mathcal{P}^{ext} = \int_{\partial\Omega} \langle t^{ext}, \delta u \rangle + \int_{\partial\Omega} \langle g^{ext}, D^1(\delta u) \rangle,$$

where $D^1(\delta u)$ is a suitable first order space differential operator, we say, by extension of the former definition, that t^{ext} and g^{ext} are strongly and weakly independent tractions whenever they are the conjugates of strongly or weakly independent surface vector fields δu and $D^1(\delta u)$.

An example of strong and weak independence which will be pertinent in the framework of the present paper is that which can be established between the displacement assigned on the surface and its curl or, alternatively, its normal derivative. Indeed, if for convenience we choose an orthonormal local basis $\{\tau, \nu, n\}$ on S , where τ and ν are unit tangent vectors to S , while n is the outward unit normal vector, we can recognize that

$$\nabla u \cdot n = \left(\frac{\partial u_\tau}{\partial x_n}, \frac{\partial u_\nu}{\partial x_n}, \frac{\partial u_n}{\partial x_n} \right)^T \quad \text{and} \quad \text{curl } u = \left(\frac{\partial u_n}{\partial x_\nu} - \frac{\partial u_\nu}{\partial x_n}, \frac{\partial u_\tau}{\partial x_n} - \frac{\partial u_n}{\partial x_\tau}, \frac{\partial u_\nu}{\partial x_\tau} - \frac{\partial u_\tau}{\partial x_\nu} \right)^T, \quad (1.21)$$

where we clearly indicated by u_τ , u_ν and u_n the components of the displacement field in such local basis and x_τ , x_ν and x_n the space coordinates of a generic material point in the same reference frame.

It is known that the fact of fixing the displacement field u on the surface does not fix its normal derivatives $\partial u_\tau / \partial x_n$, $\partial u_\nu / \partial x_n$, $\partial u_n / \partial x_n$, while it fixes the tangential derivatives $\partial u_n / \partial x_\nu$, $\partial u_n / \partial x_\tau$, $\partial u_\nu / \partial x_\tau$, $\partial u_\tau / \partial x_\nu$. In this optic, we can say that u and $\nabla u \cdot n$ are strongly independent surface vector fields, while u and $\text{curl } u$ are weakly independent surface vector fields. Indeed, even if when fixing the displacement u on the surface its tangential derivatives result to be fixed, the arbitrariness of the normal derivatives still allows to globally choose $\text{curl } u$ in an arbitrary way.

Therefore (regarding the $\text{curl } u$), it is impossible to fix as constant the $\text{curl } u$ but varying only u . In contrast, it is possible to fix as constant $\nabla u \cdot n$ and to vary u arbitrarily. In summary in this example

$$\begin{array}{lll} u & \text{and} & \nabla u \cdot n \\ u & \text{and} & \text{curl } u \end{array} \quad \begin{array}{l} \text{are strongly independent,} \\ \text{are (only) weakly independent.} \end{array}$$

2 Bulk equations and (Mindlin's weakly independent) boundary conditions in the indeterminate couple stress model

In this section we re-propose the results presented by Mindlin and Tiersten but explicitly using a variational procedure. More particularly, we try to explicitly present some reasonings that can be made to obtain their bulk equations and boundary conditions. Nevertheless, how we will show in due detail in the remainder of this paper, such reasonings are not in complete agreement with other sets of boundary conditions which can be assigned when dealing with a couple stress theory, and which are equally legitimate.

2.1 Equilibrium and constitutive equations of the indeterminate couple stress model

Since we consider that the solution belongs to $H^1(\Omega)$, we take free variations $\delta u \in C^\infty(\bar{\Omega})$ of the energy

$$W(\nabla u, \nabla \text{curl } u) = W_{\text{lin}}(\nabla u) + \widetilde{W}_{\text{curv}}(\nabla \text{curl } u), \quad (2.1)$$

where

$$\begin{aligned} W_{\text{lin}}(\nabla u) &= \mu \|\text{sym } \nabla u\|^2 + \frac{\lambda}{2} [\text{tr}(\nabla u)]^2 = \mu \|\text{dev sym } \nabla u\|^2 + \frac{2\mu + 3\lambda}{6} [\text{tr}(\nabla u)]^2, \\ \widetilde{W}_{\text{curv}}(\nabla \text{curl } u) &= \alpha_1 \|\text{dev sym } \nabla [\text{axl}(\text{skew } \nabla u)]\|^2 + \alpha_2 \|\text{skew } \nabla [\text{axl}(\text{skew } \nabla u)]\|^2, \end{aligned} \quad (2.2)$$

and we obtain the first variation of the action functional as

$$\begin{aligned} \delta \mathcal{A} = \delta \int_{\Omega} -W(\nabla u, \nabla \text{axl}(\text{skew } \nabla u)) dv = & - \int_{\Omega} \left[2\mu \langle \text{sym } \nabla u, \text{sym } \nabla \delta u \rangle + \lambda \text{tr}(\nabla u) \text{tr}(\nabla \delta u) \right. \\ & + 2\alpha_1 [\langle \text{dev sym } \nabla [\text{axl}(\text{skew } \nabla u)], \text{dev sym } \nabla [\text{axl}(\text{skew } \nabla \delta u)] \rangle] \\ & \left. + 2\alpha_2 \langle \text{skew } \nabla [\text{axl}(\text{skew } \nabla u)], \text{skew } \nabla [\text{axl}(\text{skew } \nabla \delta u)] \rangle \right] dv. \end{aligned} \quad (2.3)$$

Or equivalently, applying the classical divergence theorem

$$\begin{aligned} \delta \mathcal{A} = & \int_{\Omega} \langle \text{Div}(2\mu \text{sym } \nabla u + \lambda \text{tr}(\nabla u)\mathbf{1}), \delta u \rangle dv - \int_{\partial\Omega} \langle (2\mu \text{sym } \nabla u + \lambda \text{tr}(\nabla u)\mathbf{1}) \cdot n, \delta u \rangle da \\ & - 2 \int_{\Omega} \left\langle \alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)], \nabla \text{axl}(\text{skew } \nabla \delta u) \right\rangle dv. \end{aligned} \quad (2.4)$$

The classical divergence theorem can be applied again to the last term appearing in the previous equation. In particular, we can notice that the following chain of equalities holds

$$\begin{aligned} & \int_{\Omega} \left\langle \alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)], \nabla \text{axl}(\text{skew } \nabla \delta u) \right\rangle dv \\ &= \int_{\Omega} \left\langle -\text{Div} \left(\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)] \right), \text{axl}(\text{skew } \nabla \delta u) \right\rangle dv \\ & \quad + \int_{\partial\Omega} \left\langle \left[\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)] \right] \cdot n, \text{axl}(\text{skew } \nabla \delta u) \right\rangle da \\ &= \frac{1}{2} \int_{\Omega} \left\langle -\text{anti}\{\text{Div}(\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)])\}, \text{skew } \nabla \delta u \right\rangle dv \\ & \quad + \int_{\partial\Omega} \left\langle \left[\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)] \right] \cdot n, \text{axl}[\text{skew } \nabla \delta u] \right\rangle da \\ &= \frac{1}{2} \int_{\Omega} \left\langle -\text{skew anti}\{\text{Div}(\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)])\}, \nabla \delta u \right\rangle dv \\ & \quad + \int_{\partial\Omega} \left\langle \left[\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)] \right] \cdot n, \text{axl}(\text{skew } \nabla \delta u) \right\rangle da \\ &= \frac{1}{2} \int_{\Omega} \left\langle \text{Div}[\text{skew anti}\{\text{Div}(\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)])\}], \delta u \right\rangle dv \\ & \quad - \frac{1}{2} \int_{\partial\Omega} \left\langle [\text{skew anti}\{\text{Div}(\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)])\}] \cdot n, \delta u \right\rangle da \\ & \quad + \int_{\partial\Omega} \left\langle \left[\alpha_1 \text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)] + \alpha_2 \text{skew}[\nabla \text{axl}(\text{skew } \nabla u)] \right] \cdot n, \text{axl}(\text{skew } \nabla \delta u) \right\rangle da, \end{aligned}$$

where n is the unit outward normal vector at the surface $\partial\Omega$. Hence, recalling that $\text{skew anti}(\cdot) = \text{anti}(\cdot)$, the first variation (2.4) of the action functional can be finally rewritten as

$$\delta \mathcal{A} = \int_{\Omega} \langle \text{Div}(\sigma - \tilde{\tau}), \delta u \rangle dv - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle dv - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da,$$

for all virtual displacements $\delta u \in C^\infty(\bar{\Omega})$, where σ is the symmetric local force-stress tensor

$$\sigma = 2\mu \text{sym } \nabla u + \lambda \text{tr}(\nabla u)\mathbf{1} \in \text{Sym}(3), \quad (2.5)$$

$\tilde{\tau}$ represents the second order nonlocal force-stress tensor (which here is automatically skew-symmetric)

$$\begin{aligned}
\tilde{\tau} &= \alpha_1 \text{ anti Div}(\text{dev sym}[\nabla \text{axl}(\text{skew } \nabla u)]) + \alpha_2 \text{ anti Div}(\text{skew}[\nabla \text{axl}(\text{skew } \nabla u)]) \\
&= \frac{\alpha_1}{2} \text{ anti Div}(\text{dev sym}[\nabla \text{curl } u]) + \frac{\alpha_2}{2} \text{ anti Div}(\text{skew}[\nabla \text{curl } u]) \\
&= \text{anti Div} \left[\frac{\alpha_1}{2} \text{dev sym}(\nabla \text{curl } u) + \frac{\alpha_2}{2} \text{skew}(\nabla \text{curl } u) \right] \\
&= \frac{1}{2} \text{anti Div} [\tilde{m}] \in \mathfrak{so}(3),
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\tilde{m} &= 2\alpha_1 \text{dev sym}(\nabla \text{axl}(\text{skew } \nabla u)) + 2\alpha_2 \text{skew}(\nabla \text{axl}(\text{skew } \nabla u)) \\
&= [\alpha_1 \text{dev sym}(\nabla \text{curl } u) + \alpha_2 \text{skew}(\nabla \text{curl } u)] \\
&= \left[\frac{\alpha_1 + \alpha_2}{2} \nabla \text{curl } u + \frac{\alpha_1 - \alpha_2}{2} [\nabla \text{curl } u]^T \right]
\end{aligned} \tag{2.7}$$

is the second order hyperstress tensor (couple stress) which may or may not be symmetric, depending on the material parameters. The asymmetry of force stress is a hidden constitutive assumption, compared to the development in [30]. Postulating a particular form of the power of external actions, the equilibrium equation can therefore be written as

$$\text{Div } \tilde{\sigma}_{\text{total}} + f^{\text{ext}} = 0, \tag{2.8}$$

where we clearly set $\tilde{\sigma}_{\text{total}} = \sigma - \tilde{\tau}$.

2.2 Classical (weakly independent) boundary conditions in the indeterminate couple stress model

In the previous section, we have shown that the power of internal actions can be finally written in compact form as

$$\mathcal{P}^{\text{int}} = \delta \mathcal{A} = \int_{\Omega} \langle \text{Div}(\sigma - \tilde{\tau}), \delta u \rangle dv - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da - \frac{1}{2} \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{curl } \delta u \rangle da. \tag{2.9}$$

Such form of the power of internal actions seems to suggest 6 possible independent prescriptions of mechanical boundary conditions; three for the normal components of the total force stress $(\sigma - \tilde{\tau}) \cdot n$ and three for the normal components of the couple stress tensor. The possible Dirichlet boundary conditions on $\Gamma \subset \partial\Omega$ would seem to be the 6 conditions²

$$u = u^{\text{ext}}, \quad \text{axl}(\text{skew } \nabla u) = \frac{1}{2} \text{curl } u = \tilde{a}^{\text{ext}} \quad (\text{or equivalently } \text{curl } u = 2\tilde{a}^{\text{ext}}), \tag{2.10}$$

for two given functions $u^{\text{ext}}, \tilde{a}^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the boundary $\Gamma \subset \partial\Omega$ (3+3 boundary conditions).

However, following Koiter [47] we must note that the following remark holds true:

Remark 2.1. Assume that $u \in C^\infty(\bar{\Omega})$ and $u|_{\Gamma}$ is known. Then $\text{curl } u|_{\Gamma}$ exists and for all open subsets $\Gamma \subset \partial\Omega$ the integral $\int_{\Gamma} \langle \text{curl } u, n \rangle da$ is already known, while $\int_{\Gamma} \langle \text{curl } u, \tau \rangle da$ is still free, where τ is any tangential vector field on $\Gamma \subset \partial\Omega$. This fact follows using Stokes' circulation theorem

$$\int_{\Gamma} \langle \text{curl } u, n \rangle da = \int_{\partial\Gamma} \langle u(\gamma(t)), \gamma'(t) \rangle ds, \tag{2.11}$$

where τ is a continuous unit vector field tangent to the curve $\partial\Gamma = \{\gamma(t) | t \in [a, b]\}$ compatible with the unit vector field n normal to the surface Γ .

This leads us to the next correct observation:

²as indeed proposed by Grioli [35] in concordance with the Cosserat kinematics for independent fields of displacements and microrotation.

Remark 2.2. *Only the two tangential components of $\text{curl } u$ may be independently prescribed on an open subset of the boundary. However, we may have six independent conditions in one point on Γ , but not on an open subset of the boundary.*

Already Mindlin and Tiersten [59] have also correctly remarked that in this formulation only 5 mechanical boundary conditions can be prescribed. Using our notations, their argument runs as follows:

$$\begin{aligned}
\frac{1}{2}\langle \tilde{m} \cdot n, \text{curl } \delta u \rangle &= \frac{1}{2}\langle (T + Q) \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle = \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle Q \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle \\
&= \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle (n \otimes n) \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle = \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle ((\text{sym } \tilde{m}) \cdot n, n) \cdot n, \text{curl } \delta u \rangle \\
&= \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle n, ((\text{sym } \tilde{m}) \cdot n, n) \text{curl } \delta u \rangle \tag{2.12} \\
&= \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle n, \text{curl } [((\text{sym } \tilde{m}) \cdot n, n) \delta u] \rangle - \frac{1}{2}\langle n, \nabla ((\text{sym } \tilde{m}) \cdot n, n) \times \delta u \rangle \\
&= \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle n, \text{curl } [((\text{sym } \tilde{m}) \cdot n, n) \delta u] \rangle - \frac{1}{2}\langle n \times \nabla ((\text{sym } \tilde{m}) \cdot n, n), \delta u \rangle \\
&= \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle n, \text{curl } [((\text{sym } \tilde{m}) \cdot n, n) \delta u] \rangle - \frac{1}{2}\langle n \times \nabla ((\text{sym } \tilde{m}) \cdot n, n), \delta u \rangle,
\end{aligned}$$

where \otimes denotes the dyadic product of two vectors, we have used the property $(\eta \otimes \xi) \cdot a = \eta \langle \xi, a \rangle$ (for vectors η, ξ and a), the formula $\text{curl } (\psi \delta u) = \nabla \psi \times \delta u + \psi \text{curl } \delta u$ (for any scalar field ψ) and the fact that $n \otimes n$ is a symmetric second order tensor. The power of internal actions (2.9) can hence be rewritten as

$$\begin{aligned}
\mathcal{P}^{\text{int}} &= \int_{\Omega} \langle \text{Div}(\sigma - \tilde{\tau}), \delta u \rangle dv - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da \\
&\quad - \int_{\partial\Omega} \left\{ \frac{1}{2}\langle T \cdot (\tilde{m} \cdot n), \text{curl } \delta u \rangle + \frac{1}{2}\langle n, \text{curl } [((\text{sym } \tilde{m}) \cdot n, n) \delta u] \rangle - \frac{1}{2}\langle n \times \nabla ((\text{sym } \tilde{m}) \cdot n, n), \delta u \rangle \right\} da.
\end{aligned}$$

It can be remarked that the second term in the last surface integral can be rewritten as a bulk integral by means of the divergence theorem, so that the power of internal actions can also be rewritten in a further equivalent form in addition to the one already established in (2.9)

$$\begin{aligned}
\mathcal{P}^{\text{int}} &= \int_{\Omega} \langle \text{Div}(\sigma - \tilde{\tau}), \delta u \rangle dv - \int_{\Omega} \frac{1}{2} \text{div} \{ \text{curl } [((\text{sym } \tilde{m}) \cdot n, n) \delta u] \} dv \tag{2.13} \\
&\quad - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \nabla ((\text{sym } \tilde{m}) \cdot n, n), \delta u \rangle da - \int_{\partial\Omega} \frac{1}{2} \langle (\tilde{m} \cdot n), T \cdot \text{curl } \delta u \rangle da,
\end{aligned}$$

where the fact that the tangential projector T is symmetric has also be used.

Mindlin and Tiersten [59] concluded that 3 boundary conditions derive from the first surface integral and two other from the second surface integral, since [59, p. 432] “the normal component of the couple stress vector $[\langle \tilde{m} \cdot n, n \rangle = \langle (\text{sym } \tilde{m}) \cdot n, n \rangle]$ on $\partial\Omega$ enters only in the combination with the force-stress vector shown in the coefficient of δu in the surface integral ...”. Indeed, Mindlin and Tiersten are assuming to assign arbitrarily the displacement and the tangential components of its curl on the surface $\partial\Omega$.

As we will deeply discuss in Section 4, this choice leads to a possible set of boundary conditions in the indeterminate couple stress model. Nevertheless, this choice is not unique and assigning at the boundary different virtual fields as the virtual displacement and its normal derivative will lead us to a set of boundary conditions that are not equivalent to those proposed by Mindlin and Tiersten.

2.2.1 Geometric (essential, or kinematical) weakly independent boundary conditions

Based on the expression (2.13) of the power of internal actions, Mindlin and Tiersten [59] concluded that the geometric boundary conditions on $\Gamma \subset \partial\Omega$ are the five independent conditions

$$u|_{\Gamma} = u^{\text{ext}}, \tag{3bc} \tag{2.14}$$

$$(\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \tilde{a}^{\text{ext}}, \tag{2bc}$$

for given functions $u^{\text{ext}}, \tilde{a}^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the portion Γ of the boundary.

An equivalent form of the above boundary condition is

$$\begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, & (3\text{bc}) \\ (\mathbb{1} - n \otimes n) \cdot \text{axl}(\text{skew } \nabla u)|_{\Gamma} &= \frac{1}{2}(\mathbb{1} - n \otimes n) \cdot \tilde{a}^{\text{ext}}, & (2\text{bc}) \end{aligned} \quad (2.15)$$

for given functions $u^{\text{ext}}, \tilde{a}^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the boundary. The latter condition prescribes only the tangential component of $\text{axl}(\text{skew } \nabla u) = \frac{1}{2} \text{curl } u$. Therefore, we may prescribe only 3+2 independent geometric boundary conditions. Regarding this formulation, an existence result was proven in [45].

In order to give a first comparison with the boundary conditions which are coming from the full strain gradient approach, let us remark that:

Lemma 2.3. [Equivalence of geometric boundary conditions] *We consider a vector field $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $u \in C^\infty(\Omega)$. The following sets of boundary conditions are equivalent:*

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} &= (\mathbb{1} - n \otimes n) \cdot \tilde{a}^{\text{ext}} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} &= (\mathbb{1} - n \otimes n) \cdot a^{\text{ext}} \end{aligned} \right. \quad (2.16)$$

in the sense that one set of boundary conditions defines completely the other set of boundary conditions, where n is the unit outward normal vector at the surface $\Gamma \subset \partial\Omega$ and \tilde{a}^{ext} and a^{ext} can be a priori related.

Proof. The proof is included in Appendix A.5. □

In summary, we can say that if purely kinematical boundary conditions are assigned in the indeterminate couple stress model, in virtue of the previous Lemma 2.3, it is equivalent to assign on one portion of the boundary the displacement and the tangential part of its curl **or** the displacement and the tangential part of its normal derivative. As we will see, things become much more complicated when one wants to deal with traction or mixed boundary conditions, since it is not straightforward to individuate the equivalence between different sets of boundary conditions which are nevertheless equally legitimate.

2.2.2 Classical (weakly independent) traction boundary conditions

Always considering the expression (2.13) of the power of internal actions, the possible traction boundary conditions on $\partial\Omega \setminus \bar{\Gamma}$ given by Mindlin and Tiersten [59] in the axial formulation are

$$\begin{aligned} (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \nabla \langle (\text{sym } \tilde{m}) \cdot n, n \rangle &= \tilde{t}^{\text{ext}}, & \text{traction} & & (3 \text{ bc}) \\ (\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n &= (\mathbb{1} - n \otimes n) \cdot \tilde{g}^{\text{ext}}, & \text{double force traction} & & (2 \text{ bc}) \end{aligned} \quad (2.17)$$

for prescribed functions $\tilde{t}^{\text{ext}}, \tilde{g}^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the portion $\partial\Omega \setminus \bar{\Gamma}$ of the boundary.

Since δu and $(\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u$ are weakly independent, at this point we are tempted to conclude that the equality

$$\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t}, \delta u \rangle da + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u \rangle da = 0 \quad (2.18)$$

for all $\delta u \in C^2(\bar{\Omega})$ does not imply that $\tilde{t}|_{\partial\Omega \setminus \bar{\Gamma}} = 0$ and $(\mathbb{1} - n \otimes n) \cdot \tilde{g}|_{\partial\Omega \setminus \bar{\Gamma}} = 0$. However, this holds true after using the Lemmas included in Appendix A.8.

We want also to explicitly point out that (2.14) and (2.17) correctly describe the maximal number of independent boundary conditions in the indeterminate couple stress model but even if these conditions have been re-derived again and again by Yang et al. [88], Park and Gao [74], [50], etc. among others they are not the only possible choice in the couple stress model. This is explained in the following two subsections. Prescribing δu and $(\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u$ on the boundary means that we have prescribed independent geometrical boundary conditions, this is also the argumentation of Mindlin and Tiersten [59], Koiter [47], Sokolowski[82], etc. However, the prescribed traction conditions are **not strongly independent** but only weakly independent in the sense established in Section 1.4. For this reason we claim that, in order to prescribe strongly independent geometric

boundary conditions and their corresponding energetic conjugate, we have to prescribe u and $(\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n$. In other words, as it is well assessed in the framework of full second gradient theories (see also the expression of the external power given in (3.8)), we prescribe on the boundary the following part of the power of external actions

$$\int_{\partial\Omega} \langle \tilde{t}^{\text{ext}}, \delta u \rangle da + \int_{\partial\Omega} \langle \tilde{g}^{\text{ext}}, (\mathbb{1} - n \otimes n) \cdot \nabla \delta u \cdot n \rangle da, \quad (2.19)$$

in which now δu and $(\mathbb{1} - n \otimes n) \cdot \nabla \delta u \cdot n$ are strongly independent and \tilde{g}^{ext} does not produce (anymore hidden) work against δu . This type of strongly independent boundary conditions are also correctly considered already by Bleustein [9], but for the full strain gradient elasticity case only. We give more details in the following section.

3 Through second gradient elasticity towards the indeterminate couple stress theory: a direct approach

Independently of the method that one wants to choose to set up the correct set of bulk equations and associated boundary conditions for the indeterminate couple stress model, such set of equations must be compatible with a variational principle based on the form (2.1) of the strain energy density. We present two different ways of performing such variational treatment: the first one passes through a full second gradient approach and the second one, which we call direct approach, is based on the fact that the curvature energy is regarded as a function of the second order tensor $\nabla \text{curl } u$ instead than of the third order tensor $\nabla \nabla u$ or $\nabla \text{sym } \nabla u$.

Worthless to say, as expected, we will find that such two approaches are equivalent and we will explicitly establish this equivalence in eqs. (3.31)-(3.34).

3.1 Second gradient model: general variational setting

In this section, we show how the couple stress model can be regarded as a particular case of the second gradient model.

3.1.1 First variation of the action functional: power of internal actions

Let us consider the second gradient strain energy density $W(\nabla u, \nabla \nabla u)$ and the associated action functional in the static case (no inertia considered here)

$$\mathcal{A} = - \int_{\Omega} W(\nabla u, \nabla \nabla u) dv.$$

The first variation of the action functional can be interpreted as the power of internal actions \mathcal{P}^{int} of the considered system and can be computed as follows

$$\mathcal{P}^{\text{int}} = \delta \mathcal{A} = - \int_{\Omega} \left(\frac{\partial W}{\partial u_{i,j}} \delta u_{i,j} + \frac{\partial W}{\partial u_{i,jk}} \delta u_{i,jk} \right) dv,$$

where we used Levi-Civita index notation together with Einstein notation of sum over repeated indices. Integrating a first time by parts and using the divergence theorem we get

$$\delta \mathcal{A} = - \int_{\partial\Omega} \frac{\partial W}{\partial u_{i,j}} n_j \delta u_i da + \int_{\Omega} \left(\frac{\partial W}{\partial u_{i,j}} \right)_{,j} \delta u_i dv - \int_{\partial\Omega} \frac{\partial W}{\partial u_{i,jk}} n_k \delta u_{i,j} da + \int_{\Omega} \left(\frac{\partial W}{\partial u_{i,jk}} \right)_{,k} \delta u_{i,j} dv.$$

Integrating again by parts the last bulk term we get

$$\delta \mathcal{A} = - \int_{\partial\Omega} \left(\frac{\partial W}{\partial u_{i,j}} - \left(\frac{\partial W}{\partial u_{i,jk}} \right)_{,k} \right) n_j \delta u_i da + \int_{\Omega} \left[\frac{\partial W}{\partial u_{i,j}} - \left(\frac{\partial W}{\partial u_{i,jk}} \right)_{,k} \right]_{,j} \delta u_i dv - \int_{\partial\Omega} \frac{\partial W}{\partial u_{i,jk}} n_k \delta u_{i,j} da,$$

which can also be rewritten as

$$\delta \mathcal{A} = - \int_{\partial\Omega} (\sigma_{ij} - \mathbf{m}_{ijk,k}) n_j \delta u_i da + \int_{\Omega} (\sigma_{ij} - \mathbf{m}_{ijk,k})_{,j} \delta u_i dv - \int_{\partial\Omega} \mathbf{m}_{ijk} n_k \delta u_{i,j} da, \quad (3.1)$$

if one sets

$$\sigma_{ij} = \frac{\partial W}{\partial u_{i,j}}, \quad \mathbf{m}_{ijk} = \frac{\partial W}{\partial u_{i,jk}},$$

or equivalently, in compact notation:

$$\sigma = \frac{\partial W}{\partial \nabla u}, \quad \mathbf{m} = \frac{\partial W}{\partial \nabla \nabla u}. \quad (3.2)$$

3.1.2 Surface integration by parts and independent variations

At this point, it must be considered that expression (3.1) can still be manipulated remarking that the tangential trace of the gradient of virtual displacement can be integrated by parts once again and that the surface divergence theorem can be applied to this tangential part of $\nabla \delta u$. Using the brief digression concerning differential geometry and recalling the properties (1.12) of the tangential and normal projectors, we can now ulteriorly manipulate the last term in eq. (3.1) as follows,

$$\begin{aligned} \int_{\partial\Omega} \mathbf{m}_{ijk} n_k \delta u_{i,j} da &= \int_{\partial\Omega} B_{ij} \delta u_{i,h} \delta_{hj} da = \int_{\partial\Omega} B_{ij} \delta u_{i,h} (T_{hj} + Q_{hj}) da \\ &= \int_{\partial\Omega} B_{ij} \delta u_{i,h} T_{hj} da + \int_{\partial\Omega} B_{ij} \delta u_{i,h} Q_{hj} da, \\ &= \int_{\partial\Omega} (T_{hj} B_{ij}) \delta u_{i,h} da + \int_{\partial\Omega} (B_{ij} n_j) \delta (u_{i,h}) n_h da \\ &= \int_{\partial\Omega} (T_{hp} T_{pj} B_{ij}) \delta u_{i,h} da + \int_{\partial\Omega} (B_{ij} n_j) \delta (u_{i,h} n_h) da, \end{aligned} \quad (3.3)$$

where we clearly set

$$B_{ij} = \mathbf{m}_{ijk} n_k. \quad (3.4)$$

We can hence recognize in the last term of this formula that the virtual variation of the normal derivative $u_{i,h} n_h =: u_i^n = (\nabla u \cdot n)_i$ of the displacement field appears. As for the first term, it can be still manipulated suitably integrating by parts and then using the surface divergence theorem (1.18), so that we can write

$$\begin{aligned} \int_{\partial\Omega} \mathbf{m}_{ijk} n_k \delta u_{i,j} da &= \int_{\partial\Omega} T_{hp} \left[(T_{pj} B_{ij} \delta u_i)_{,h} - (T_{pj} B_{ij})_{,h} \delta u_i \right] da + \int_{\partial\Omega} (B_{ij} n_j) \delta u_i^n da, \\ &= \int_{\partial\Gamma} \llbracket B_{ip} \nu_p \delta u_i \rrbracket ds - \int_{\partial\Omega} (T_{pj} B_{ij})_{,h} T_{hp} \delta u_i da + \int_{\partial\Omega} (B_{ij} n_j) \delta u_i^n da. \end{aligned} \quad (3.5)$$

The final variation of the second gradient action functional given in (3.1), can therefore be written as

$$\begin{aligned} \mathcal{P}^{\text{int}} = \delta \mathcal{A} &= \int_{\Omega} (\sigma_{ij} - \mathbf{m}_{ijk,k})_{,j} \delta u_i dv - \int_{\partial\Omega} \left[(\sigma_{ij} - \mathbf{m}_{ijk,k}) n_j - (T_{pj} \mathbf{m}_{ijk} n_k)_{,h} T_{hp} \right] \delta u_i da \\ &\quad - \int_{\partial\Omega} (\mathbf{m}_{ijk} n_k n_j) \delta u_i^n da - \int_{\partial\Gamma} \llbracket \mathbf{m}_{ijk} n_k \nu_j \delta u_i \rrbracket ds, \end{aligned} \quad (3.6)$$

or equivalently in compact notation

$$\begin{aligned} \mathcal{P}^{\text{int}} = \delta \mathcal{A} &= \int_{\Omega} \langle \text{Div} (\sigma - \text{Div } \mathbf{m}), \delta u \rangle dv - \int_{\partial\Omega} \langle (\sigma - \text{Div } \mathbf{m}) \cdot n - \nabla [(\mathbf{m} \cdot n) \cdot T] : T, \delta u \rangle da \\ &\quad - \int_{\partial\Omega} \langle (\mathbf{m} \cdot n) \cdot n, (\nabla \delta u) \cdot n \rangle da - \int_{\partial\Gamma} \llbracket \langle (\mathbf{m} \cdot n) \cdot \nu, \delta u \rangle \rrbracket ds. \end{aligned} \quad (3.7)$$

If now one recalls the principle of virtual powers according to which a given system is in equilibrium if the power of internal forces is equal to the power of external forces, it is straightforward that the expression (3.6) naturally suggests which is the correct expression for the power of external forces that a second gradient continuum may sustain, namely:

$$\mathcal{P}^{\text{ext}} = \int_{\Omega} f_j^{\text{ext}} \delta u_j dv + \int_{\partial\Omega} t_j^{\text{ext}} \delta u_j da + \int_{\partial\Omega} g_j^{\text{ext}} \delta u_j^n da + \int_{\partial\Gamma} \llbracket \pi_j^{\text{ext}} \delta u_j \rrbracket ds, \quad (3.8)$$

where f^{ext} are external bulk forces (expending power on displacement), t^{ext} are external surface forces (expending power on displacement), g^{ext} are external surface double-forces (expending power on the normal derivative of displacement) and π^{ext} are external line forces (expending power on displacement). Imposing that

$$\mathcal{P}^{\text{int}} + \mathcal{P}^{\text{ext}} = 0 \quad (3.9)$$

and localizing, one can get the strong form of the equations of motion and associated boundary conditions for a second gradient continuum.

Therefore, the equilibrium equation for a second gradient continuum is

$$\text{Div}(\sigma - \text{Div} \mathbf{m}) + f^{\text{ext}} = 0. \quad (3.10)$$

This set of partial differential equation can be complemented with the following boundary conditions:

- **Strongly independent, second gradient, geometric boundary conditions**

$$\begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, & (3\text{bc}) \\ \nabla u \cdot n|_{\Gamma} &= a^{\text{ext}}, & (3\text{bc}) \end{aligned} \quad (3.11)$$

for given functions $u^{\text{ext}}, a^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the portion Γ of the boundary.

- **Strongly independent, second gradient, traction boundary conditions**

Traction boundary condition on $\partial\Omega \setminus \bar{\Gamma}$:

$$\begin{aligned} (\sigma - \text{Div} \mathbf{m}) \cdot n - \nabla[(\mathbf{m} \cdot n) \cdot T] : T|_{\partial\Omega \setminus \bar{\Gamma}} &= t^{\text{ext}}, & \text{traction} & & (3 \text{ bc}) \\ (\mathbf{m} \cdot n) \cdot n|_{\partial\Omega \setminus \bar{\Gamma}} &= g^{\text{ext}}, & \text{"double force traction"} & & (3 \text{ bc}) \end{aligned} \quad (3.12)$$

for prescribed functions $t^{\text{ext}}, g^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the boundary.

Traction boundary condition on $\partial\Gamma$:

$$[(\mathbf{m} \cdot n) \cdot \nu]|_{\partial\Gamma} = \pi^{\text{ext}} \quad \text{"line force traction"} \quad (3 \text{ bc}) \quad (3.13)$$

for a prescribed function $\pi^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on $\partial\Gamma$.

We want to stress the fact that in the framework of a second gradient theory the test functions that can be arbitrarily assigned on the boundary $\partial\Omega$ are the virtual displacement δu and the normal derivative of the virtual displacement $\nabla(\delta u) \cdot n$. This means that one has $3 + 3 = 6$ independent geometric boundary conditions that can be assigned on the boundary of the considered second gradient medium. Analogously one can think to assign $3 + 3 = 6$ traction conditions on the force (in duality of δu) and double force (in duality of $\nabla(\delta u) \cdot n$) respectively. Hence, in a complete second gradient theory 6 independent scalar conditions must be assigned on the boundary in order to have a well-posed problem.

3.2 The indeterminate couple stress model viewed as a subclass of the second gradient elasticity model

As in the previous case, we consider a particular strain energy density of the type

$$W = W_{\text{lin}}(\nabla u) + W_{\text{curv}}(\nabla[\text{axl}(\text{skew} \nabla u)]) = W_{\text{lin}}(\nabla u) + \widetilde{W}_{\text{curv}}(\nabla \text{curl} u),$$

where $W_{\text{lin}}(\nabla u)$ is given in eq. (2.2), while the curvature energy $\widetilde{W}_{\text{curv}}(\nabla \text{curl} u)$ also discussed in the previous section, is given by

$$\begin{aligned} \widetilde{W}_{\text{curv}}(\nabla \text{curl} u) &= \frac{\alpha_1}{4} \|\text{sym} \nabla \text{curl} u\|^2 + \frac{\alpha_2}{4} \|\text{skew} \nabla \text{curl} u\|^2 \\ &=: \frac{\alpha_1}{4} \|S\|^2 + \frac{\alpha_2}{4} \|A\|^2 = \frac{\alpha_1}{4} S_{lm} S_{lm} + \frac{\alpha_2}{4} A_{lm} A_{lm}, \end{aligned} \quad (3.14)$$

where we set

$$S_{pq} := (\text{sym } \nabla \text{curl } u)_{pq} = \frac{\epsilon_{prs} u_{s,rq} + \epsilon_{qrs} u_{s,rp}}{2}, \quad A_{pq} := (\text{skew } \nabla \text{curl } u)_{pq} = \frac{\epsilon_{prs} u_{s,rq} - \epsilon_{qrs} u_{s,rp}}{2}. \quad (3.15)$$

This decomposition of the curvature energy is equivalent to that which can be found in Mindlin and Tiersten [59] and presented by us in eq. (1.8).

Regarding the curvature energy (3.14) as a particular case of second gradient energy, we can directly calculate the particular form of the third order hyperstress tensor as

$$\tilde{\mathbf{m}}_{ijk} = \frac{\partial \tilde{W}_{\text{curv}}}{\partial u_{i,jk}} = \frac{\partial \tilde{W}_{\text{curv}}}{\partial S_{pq}} \frac{\partial S_{pq}}{\partial u_{i,jk}} + \frac{\partial \tilde{W}_{\text{curv}}}{\partial A_{pq}} \frac{\partial A_{pq}}{\partial u_{i,jk}} = \frac{\alpha_1}{2} S_{pq} \frac{\partial S_{pq}}{\partial u_{i,jk}} + \frac{\alpha_2}{2} A_{pq} \frac{\partial A_{pq}}{\partial u_{i,jk}}. \quad (3.16)$$

It can be checked that, from (3.15), one gets

$$\frac{\partial S_{pq}}{\partial u_{i,jk}} = \frac{1}{2} (\epsilon_{pji} \delta_{qk} + \epsilon_{qji} \delta_{pk}), \quad \frac{\partial A_{pq}}{\partial u_{i,jk}} = \frac{1}{2} (\epsilon_{pji} \delta_{qk} - \epsilon_{qji} \delta_{pk}).$$

Replacing these expressions in (3.16), using the definitions (3.15) together with the identities

$$\epsilon_{pji} \epsilon_{prs} = \delta_{jr} \delta_{is} - \delta_{js} \delta_{ir}$$

and

$$\epsilon_{pji} \epsilon_{krs} = \delta_{pk} \delta_{jr} \delta_{is} - \delta_{pk} \delta_{js} \delta_{ir} - \delta_{pr} \delta_{jk} \delta_{is} + \delta_{pr} \delta_{js} \delta_{ik} + \delta_{ps} \delta_{jk} \delta_{ir} - \delta_{ps} \delta_{jr} \delta_{ik},$$

the fact that $S_{pk} = S_{kp}$ and $A_{pk} = -A_{kp}$ and simplifying gives

$$\begin{aligned} \tilde{\mathbf{m}}_{ijk} &= \frac{\alpha_1}{4} (\epsilon_{pji} S_{pk} + \epsilon_{qji} S_{kq}) + \frac{\alpha_2}{4} (\epsilon_{pji} A_{pk} - \epsilon_{qji} A_{kq}) = \frac{\alpha_1}{2} \epsilon_{pji} S_{pk} + \frac{\alpha_2}{2} \epsilon_{pji} A_{pk} \\ &= \frac{\alpha_1}{4} \epsilon_{pji} (\epsilon_{prs} u_{s,rk} + \epsilon_{krs} u_{s,rp}) + \frac{\alpha_2}{4} \epsilon_{pji} (\epsilon_{prs} u_{s,rk} - \epsilon_{krs} u_{s,rp}) \\ &= \frac{\alpha_1}{2} (u_{i,jk} - u_{j,ik}) + \frac{1}{4} (\alpha_1 - \alpha_2) [u_{p,ip} \delta_{jk} - u_{i,pp} \delta_{jk} + u_{j,pp} \delta_{ik} - u_{p,jp} \delta_{ik}]. \end{aligned} \quad (3.17)$$

Such particular expression of the third order hyperstress tensor can be also written in compact form as

$$\begin{aligned} \tilde{\mathbf{m}} &= \alpha_1 \nabla [\text{skew}(\nabla u)] + \frac{1}{4} (\alpha_1 - \alpha_2) [\nabla (\text{div } u) \otimes \mathbf{1} - \text{Div}(\nabla u) \otimes \mathbf{1}] \\ &\quad + \frac{1}{4} (\alpha_1 - \alpha_2) [(\text{Div}(\nabla u) \otimes \mathbf{1})^{T^{12}} - (\nabla(\text{div } u) \otimes \mathbf{1})^{T^{12}}], \end{aligned} \quad (3.18)$$

where we denote by the superscript T^{12} the transposition over the two first indices of the considered third order tensors. With such definition of the third order hyperstress tensor $\tilde{\mathbf{m}}$ one can now write the principle of virtual powers for the considered particular case in the form

$$\begin{aligned} \int_{\Omega} (\sigma_{ij} - \tilde{\mathbf{m}}_{ijk,k})_{,j} \delta u_i dv - \int_{\partial\Omega} [(\sigma_{ij} - \tilde{\mathbf{m}}_{ijk,k}) n_j - (T_{pj} \tilde{\mathbf{m}}_{ijk} n_k)_{,h} T_{hp}] \delta u_i da \\ - \int_{\partial\Omega} (\tilde{\mathbf{m}}_{ijk} n_k n_j) \delta u_i^n da - \int_{\partial\Gamma} [\tilde{\mathbf{m}}_{ijk} n_k \nu_j \delta u_i] ds \\ = - \int_{\Omega} f_i^{\text{ext}} \delta u_i dv - \int_{\partial\Omega} t_i^{\text{ext}} \delta u_i da - \int_{\partial\Omega} m_i^{\text{ext}} \delta u_i^n da - \int_{\partial\Gamma} \pi_j^{\text{ext}} \delta u_j ds. \end{aligned} \quad (3.19)$$

- We have to remark that the term

$$\begin{aligned} (\tilde{\mathbf{m}}_{ijk} n_k n_j) \delta u_i^n &= \left[\frac{\alpha_1}{2} (u_{i,jk} - u_{j,ik}) n_k n_j \right. \\ &\quad \left. + \frac{1}{4} (\alpha_1 - \alpha_2) (u_{p,ip} n_j n_j - u_{i,pp} n_j n_j + u_{j,pp} n_j n_i - u_{p,jp} n_j n_i) \right] \delta u_i^n \end{aligned} \quad (3.20)$$

is vanishing for some particular choices of the indices. In particular, if, for the sake of simplicity, one considers the introduced quantities to be all expressed in the local orthonormal basis $\{n, \tau, \nu\}$, then the aforementioned term can be rewritten as

$$(\tilde{\mathbf{m}}_{ijk} n_k n_j) \delta u_i^n = \left[\frac{\alpha_1}{2} (u_{i,11} - u_{1,i1}) + \frac{1}{4} (\alpha_1 - \alpha_2) (u_{p,ip} - u_{i,pp} + u_{1,pp} n_i - u_{p,1p} n_i) \right] \delta u_i^n.$$

It can be easily checked that such term is vanishing when $i = 1$. More precisely, we are saying that the normal component of the normal derivative δu_i^n does not contribute to the power of internal forces when considering the indeterminate couple stress model. This is equivalent to say that indeed only 2 geometric boundary conditions can be imposed on the normal derivative of virtual displacement or, equivalently, on its “traction” counterpart which is the double force.

Hence, the governing equations of the considered system can also be formally written in the form

$$\text{Div}(\sigma - \text{Div } \tilde{\mathbf{m}}) + f^{\text{ext}} = 0, \quad \text{in duality of } \delta u \quad (3.21)$$

together with the following boundary conditions induced by (3.7)³:

- **Strongly independent, geometric boundary conditions for the couple stress model on Γ (as derived by a full-gradient model)**

$$\begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ T \cdot \nabla u \cdot n|_{\Gamma} &= T \cdot a^{\text{ext}}, \end{aligned} \quad (3.22)$$

for given functions $u^{\text{ext}}, a^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the boundary.

- **Strongly independent, traction boundary conditions on $\partial\Omega \setminus \bar{\Gamma}$ (as derived by a full-gradient model)**

$$\begin{aligned} (\sigma - \text{Div } \tilde{\mathbf{m}}) \cdot n - \nabla [(\tilde{\mathbf{m}} \cdot n) \cdot T] : T &= t^{\text{ext}}, & \text{in duality of } \delta u \\ T \cdot [(\tilde{\mathbf{m}} \cdot n) \cdot n] &= T \cdot g^{\text{ext}}, & \text{in duality of } T \cdot (\nabla \delta u) \cdot n \end{aligned} \quad (3.23)$$

for prescribed functions $t^{\text{ext}}, g^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the boundary.

Traction boundary condition on the curve $\partial\Gamma$:

$$[(\tilde{\mathbf{m}} \cdot n) \cdot \nu] = \pi^{\text{ext}}, \quad (3.24)$$

for a prescribed function $\pi^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on $\partial\Gamma$.

3.3 Reduction from the third order hyperstress tensor $\tilde{\mathbf{m}}$ to Mindlin’s second order couple stress tensor $\tilde{\mathbf{m}}$

We want to prove here that the equations (3.10) and the traction boundary conditions (3.22)-(3.23) can be equivalently rewritten using Mindlin’s second order couple stress tensor

$$\begin{aligned} \tilde{\mathbf{m}} &= \frac{\alpha_1 + \alpha_2}{2} \nabla \text{curl } u + \frac{\alpha_1 - \alpha_2}{2} (\nabla \text{curl } u)^T & \text{Mindlin} \\ &= \alpha_1 \text{dev sym}(\nabla \text{curl } u) + \alpha_2 \text{skew}(\nabla \text{curl } u) & \text{equivalent form 1} \\ &= 2 \alpha_1 \text{dev sym}(\nabla \text{axl}(\text{skew } \nabla u)) + 2 \alpha_2 \text{skew}(\nabla \text{axl}(\text{skew } \nabla u)), & \text{equivalent form 2} \\ \tilde{m}_{il} &= \frac{\alpha_1 + \alpha_2}{2} \epsilon_{ijk} u_{k,jl} + \frac{\alpha_1 - \alpha_2}{2} \epsilon_{mjk} u_{k,ji}, & \text{index format} \end{aligned} \quad (3.25)$$

³We recall that in the considered couple stress model expressed in the framework of a full second gradient theory the constitutive form for m is given in eq. (3.17) or equivalently (3.18).

instead of the third order tensor given in eq. (3.18). Such second order hyper-stress tensor has been introduced by Mindlin and Tiersten [59] and we have shown in a previous section that it can be obtained by means of a direct variational approach that does not need the introduction of the third order couple stress tensor $\tilde{\mathbf{m}}$ (see eq. (2.7)).

In order to be able to set up such equivalence, we have to remark that, for the choices (3.18) and (3.25) of $\tilde{\mathbf{m}}$ and \tilde{m} , the following properties are verified (see Appendix A.2 for detailed calculations)⁴

$$\text{Div} \underbrace{\tilde{\mathbf{m}}}_{\mathbb{R}^{3 \times 3 \times 3}} = \frac{1}{2} \text{anti Div} \underbrace{\tilde{m}}_{\mathbb{R}^{3 \times 3}} = \frac{\alpha_1 + \alpha_2}{2} \Delta(\text{skew} \nabla u), \quad (3.26)$$

and

$$\nabla [(\tilde{\mathbf{m}} \cdot n) \cdot T] : T = \frac{1}{2} \nabla [\text{anti}(\tilde{m} \cdot n) \cdot T] : T, \quad (3.27)$$

$$T \cdot [(\tilde{\mathbf{m}} \cdot n) \cdot n] = \frac{1}{2} T \cdot \text{anti}(\tilde{m} \cdot n) \cdot n, \quad (3.28)$$

$$\llbracket (\tilde{\mathbf{m}} \cdot n) \cdot \nu \rrbracket = \frac{1}{2} \llbracket [\text{anti}(\tilde{m} \cdot n) \cdot n] \cdot \nu \rrbracket, \quad (3.29)$$

where

$$\tilde{\mathbf{m}} \cdot n = \frac{1}{2} \text{anti}(\tilde{m} \cdot n) = \alpha_1 [\nabla(\text{skew} \nabla u)] \cdot n + \frac{(\alpha_1 - \alpha_2)}{2} \text{skew} [\nabla(\text{Div} u) \otimes n - \text{Div}(\nabla u) \otimes n]. \quad (3.30)$$

Clearly, based upon such relationships, we can recognize the following equivalent forms for the bulk equations

$$\text{Div}(\sigma - \text{Div} \tilde{\mathbf{m}}) + f^{\text{ext}} = 0 \quad \Leftrightarrow \quad \text{Div} \left(\sigma - \frac{1}{2} \text{anti Div} \tilde{m} \right) + f^{\text{ext}} = 0, \quad (3.31)$$

in duality of δu

together with the following equivalent forms of the traction boundary conditions

$$(\sigma - \text{Div} \tilde{\mathbf{m}}) \cdot n - \nabla [(\tilde{\mathbf{m}} \cdot n) \cdot T] : T = t^{\text{ext}} \quad \Leftrightarrow \quad \left(\sigma - \frac{1}{2} \text{anti Div} \tilde{m} \right) \cdot n - \frac{1}{2} \nabla [\text{anti}(\tilde{m} \cdot n) \cdot T] : T = t^{\text{ext}} \quad (3.32)$$

in duality of δu ,

$$T \cdot [(\tilde{\mathbf{m}} \cdot n) \cdot n] = T \cdot g^{\text{ext}} \quad \Leftrightarrow \quad \frac{1}{2} T \cdot \text{anti}(\tilde{m} \cdot n) \cdot n = T \cdot g^{\text{ext}} \quad (3.33)$$

in duality of $T \cdot (\nabla \delta u) \cdot n$,

and finally the following equivalent conditions on the boundary of the boundary $\partial \Gamma$ where traction is assigned

$$\llbracket (\tilde{\mathbf{m}} \cdot n) \cdot \nu \rrbracket = \pi^{\text{ext}} \quad \Leftrightarrow \quad \frac{1}{2} \llbracket [\text{anti}(\tilde{m} \cdot n) \cdot n] \cdot \nu \rrbracket = \pi^{\text{ext}} \quad (3.34)$$

in duality of δu .

3.4 A direct way to obtain strongly independent boundary conditions in the indeterminate couple model

Let us consider again the energy

$$W = W_{\text{lin}}(\nabla u) + \widetilde{W}_{\text{curv}}(\nabla \text{curl} u),$$

with W_{lin} and $\widetilde{W}_{\text{curv}}$ defined in equations (2.2) and for which different equivalent forms of the curvature energy have been given in eq. (1.8). To the sake of completeness, we derive in this section the equations of motion and

⁴Using a classical notation $\Delta = \text{Div} \nabla$ is the Laplacian operator.

associated boundary conditions of the couple stress model by directly computing the first variation of the action functional associated to the considered energy, without noticing that such energy is indeed a very particular case of a second gradient energy. This procedure follows what was done by Mindlin and Tiersten [59] and it is presented in Section 2. Here, as done in [59], the curvature energy is regarded as a function of the second order tensor $\nabla \text{curl } u$, instead that of the third order tensor $\nabla \nabla u$. The difference of the calculation that we present here, with respect to what is done by Mindlin and Tiersten, is that we proceed further in the process of integration by parts up to the point of getting strongly independent quantities on the boundary. As it is shown in Section 3.3, the two approaches can be considered to be finally equivalent, provided that a suitable identification of the second and third order tensors appearing in the governing equations is performed.

As usual, the power of internal actions is given by the first variation of the action functional which can be directly computed as

$$\begin{aligned} \mathcal{P}^{\text{int}} &= \delta \mathcal{A} = -\delta \int_{\Omega} \left[W_{\text{lin}}(\nabla u) + \widetilde{W}_{\text{curv}}(\nabla \text{curl } u) \right] dv \\ &= -\int_{\Omega} \left\langle \frac{\partial W_{\text{lin}}}{\partial \nabla u}, \delta \nabla u \right\rangle dv - \int_{\Omega} \left\langle \frac{\partial \widetilde{W}_{\text{curv}}}{\partial S}, \delta S \right\rangle dv - \int_{\Omega} \left\langle \frac{\partial \widetilde{W}_{\text{curv}}}{\partial A}, \delta A \right\rangle dv \\ &= -\int_{\Omega} \frac{\partial W_{\text{lin}}}{\partial u_{i,j}} \delta u_{i,j} dv - \int_{\Omega} \frac{\partial \widetilde{W}_{\text{curv}}}{\partial S_{ij}} \delta S_{ij} dv - \int_{\Omega} \frac{\partial \widetilde{W}_{\text{curv}}}{\partial A_{ij}} \delta A_{ij} dv. \end{aligned} \quad (3.35)$$

Using the expression of $\widetilde{W}_{\text{curv}}$ given in (3.14) and then the definitions (3.15) for S and A together with the properties of Levi-Civita symbols, it can be checked that

$$\begin{aligned} & -\int_{\Omega} \frac{\partial \widetilde{W}_{\text{curv}}}{\partial S_{ij}} \delta S_{ij} dv - \int_{\Omega} \frac{\partial \widetilde{W}_{\text{curv}}}{\partial A_{ij}} \delta A_{ij} dv \\ &= -\frac{\alpha_1}{4} \int_{\Omega} S_{ij} \delta S_{ij} dv - \frac{\alpha_2}{4} \int_{\Omega} A_{ij} \delta A_{ij} dv = \\ &= -\frac{\alpha_1}{8} \int_{\Omega} (\epsilon_{ipq} u_{q,pj} + \epsilon_{jpp} u_{q,pi}) (\epsilon_{irs} \delta u_{s,rj} + \epsilon_{jrs} \delta u_{s,ri}) dv \\ &\quad - \frac{\alpha_2}{8} \int_{\Omega} (\epsilon_{ipq} u_{q,pj} - \epsilon_{jpp} u_{q,pi}) (\epsilon_{irs} \delta u_{s,rj} - \epsilon_{jrs} \delta u_{s,ri}) dv \\ &= -\frac{(\alpha_1 + \alpha_2)}{8} \int_{\Omega} (\epsilon_{ipq} \epsilon_{irs} u_{q,pj} \delta u_{s,rj} + \epsilon_{jpp} \epsilon_{jrs} u_{q,pi} \delta u_{s,ri}) dv \\ &\quad - \frac{(\alpha_1 - \alpha_2)}{8} \int_{\Omega} (\epsilon_{jpp} \epsilon_{irs} u_{q,pi} \delta u_{s,rj} + \epsilon_{ipq} \epsilon_{jrs} u_{q,pj} \delta u_{s,ri}) dv \\ &= -\frac{(\alpha_1 + \alpha_2)}{4} \int_{\Omega} [(\delta_{pr} \delta_{qs} - \delta_{ps} \delta_{qr}) u_{q,pi} \delta u_{s,ri}] dv \\ &\quad - \frac{(\alpha_1 - \alpha_2)}{4} \int_{\Omega} ((\delta_{ij} \delta_{pr} \delta_{qs} - \delta_{ij} \delta_{ps} \delta_{qr} - \delta_{jr} \delta_{pi} \delta_{qs} + \delta_{jr} \delta_{ps} \delta_{qi} + \delta_{js} \delta_{pi} \delta_{qr} - \delta_{js} \delta_{pr} \delta_{qi}) u_{q,pi} \delta u_{s,rj}) dv \\ &= -\frac{(\alpha_1 + \alpha_2)}{4} \int_{\Omega} (u_{s,r} - u_{r,s})_{,i} \delta u_{s,ri} dv \\ &\quad - \frac{(\alpha_1 - \alpha_2)}{4} \int_{\Omega} \left((u_{s,r} - u_{r,s})_{,i} \delta u_{s,ri} + (u_{i,si} - u_{s,ii}) \delta u_{s,rr} + (u_{r,ii} - u_{i,ri}) \delta u_{s,rs} \delta u_{s,rs} \right) dv \\ &= -\frac{\alpha_1}{2} \int_{\Omega} (u_{s,r} - u_{r,s})_{,i} \delta u_{s,ri} dv - \frac{(\alpha_1 - \alpha_2)}{4} \int_{\Omega} \left((u_{i,s} - u_{s,i})_{,i} \delta u_{s,rr} + (u_{r,i} - u_{i,r})_{,i} \delta u_{s,rs} \right) dv. \end{aligned}$$

Recalling also the results for the variation of the classical first gradient term given in (2.4) this last relation implies that the internal actions (3.35) can be rewritten as

$$\begin{aligned} \mathcal{P}^{\text{int}} &= \delta \mathcal{A} = -\int_{\Omega} (\mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}) \delta u_{i,j} dv - \frac{\alpha_1}{4} \int_{\Omega} S_{ij} \delta S_{ij} - \frac{\alpha_2}{4} \int_{\Omega} A_{ij} \delta A_{ij} dv \\ &= -\int_{\Omega} (\mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}) \delta u_{i,j} dv - \frac{\alpha_1}{2} \int_{\Omega} (u_{s,r} - u_{r,s})_{,i} \delta u_{s,ri} dv \end{aligned}$$

$$- \frac{(\alpha_1 - \alpha_2)}{4} \int_{\Omega} \left[(u_{i,s} - u_{s,i})_{,i} \delta u_{s,rr} + (u_{r,i} - u_{i,r})_{,i} \delta u_{s,rs} \right] dv.$$

Suitably integrating by parts we can hence write

$$\begin{aligned} \mathcal{P}^{\text{int}} = \delta \mathcal{A} &= - \int_{\partial \Omega} \sigma_{ij} n_j \delta u_i da + \int_{\Omega} \sigma_{ij,j} \delta u_i dv \\ &+ \int_{\Omega} \left[\frac{\alpha_1}{2} (u_{s,r} - u_{r,s})_{,ii} + \frac{(\alpha_1 - \alpha_2)}{4} \left((u_{i,s} - u_{s,i})_{,ir} + (u_{r,i} - u_{i,r})_{,is} \right) \right] \delta u_{s,r} dv \\ &- \int_{\partial \Omega} \left[\frac{\alpha_1}{2} (u_{s,r} - u_{r,s})_{,i} n_i + \frac{(\alpha_1 - \alpha_2)}{4} \left((u_{i,s} - u_{s,i})_{,i} n_r + (u_{r,i} - u_{i,r})_{,i} n_s \right) \right] \delta u_{s,r} da \\ &= - \int_{\partial \Omega} (\sigma_{ij} - \tilde{\tau}_{ij}) n_j \delta u_i da + \int_{\Omega} (\sigma_{ij} - \tilde{\tau}_{ij})_{,j} \delta u_i dv - \int_{\partial \Omega} B_{ij} \delta u_{i,j} da, \end{aligned} \quad (3.36)$$

where we set

$$\begin{aligned} \sigma_{ij} &= (\mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}), \\ \tilde{\tau}_{ij} &= \left[\frac{\alpha_1}{2} (u_{i,jpp} - u_{j,ipp}) - \frac{(\alpha_1 - \alpha_2)}{4} (u_{i,ppj} - u_{j,ppi}) \right] = \frac{(\alpha_1 + \alpha_2)}{4} (u_{i,j} - u_{j,i})_{,pp}, \end{aligned}$$

and

$$B_{ij} = \frac{\alpha_1}{2} (u_{i,j} - u_{j,i})_{,p} n_p + \frac{(\alpha_1 - \alpha_2)}{4} \left((u_{p,i} - u_{i,p})_{,p} n_j + (u_{j,p} - u_{p,j})_{,p} n_i \right). \quad (3.37)$$

With reference to eqs. (3.26) and (3.30), it can be recognized that

$$\tilde{\tau} = \text{Div } \tilde{\mathbf{m}} = \frac{1}{2} \text{anti Div } \tilde{\mathbf{m}} = \frac{\alpha_1 + \alpha_2}{2} \Delta (\text{skew } \nabla u), \quad (3.38)$$

$$B = \tilde{\mathbf{m}} \cdot \mathbf{n} = \frac{1}{2} \text{anti } (\tilde{\mathbf{m}} \cdot \mathbf{n}) = \alpha_1 [\nabla (\text{skew } \nabla u)] \cdot \mathbf{n} + \frac{(\alpha_1 - \alpha_2)}{2} \text{skew } [\nabla (\text{Div } u) \otimes \mathbf{n} - \text{Div } (\nabla u) \otimes \mathbf{n}], \quad (3.39)$$

with $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{m}}$ given in (3.18) and (3.25) respectively.

Remark 3.1. *We explicitly remark at this point (and we will point it out more precisely in the next section) that the results presented by Mindlin and Tiersten [59] are compatible with a variational procedure which stops at this point (eq. (3.36)) without proceeding further in the process of integration by parts.*

Indeed, in the view of proceeding towards the determination of strongly independent virtual variations, the last term in the expression (3.36) of the power of internal forces can still be manipulated according to the procedure (3.5) of surface integration by parts, so that one finally gets

$$\int_{\partial \Omega} B_{ij} \delta u_{i,j} da = \int_{\partial \Gamma} \llbracket B_{ij} \nu_p \delta u_i \rrbracket ds - \int_{\partial \Omega} (T_{pj} B_{ij})_{,h} T_{hp} \delta u_i da + \int_{\partial \Omega} (B_{ij} n_j) \delta u_i^n da.$$

Hence, supposing that the virtual displacement is continuous through the curves $\partial \Gamma$, the power of internal forces of the couple stress model calculated by means of a direct approach reads

$$\mathcal{P}^{\text{int}} = \int_{\Omega} (\sigma_{ij} - \tilde{\tau}_{ij})_{,j} \delta u_i dv - \int_{\partial \Omega} \left[(\sigma_{ij} - \tilde{\tau}_{ij}) n_j - (T_{pj} B_{ij})_{,h} T_{hp} \right] \delta u_i da - \int_{\partial \Omega} (B_{ij} n_j) \delta u_i^n da - \int_{\partial \Gamma} \llbracket B_{ij} \nu_p \rrbracket \delta u_i ds.$$

It has already been proven in Subsection 3.2 that only the tangent part of the normal derivative δu_i^n contributes to the power of internal actions when considering the indeterminate couple-stress model, so that the power of internal actions can be finally written as

$$\begin{aligned} \mathcal{P}^{\text{int}} &= \int_{\Omega} (\sigma_{ij} - \tilde{\tau}_{ij})_{,j} \delta u_i dv - \int_{\partial \Omega} \left[(\sigma_{ij} - \tilde{\tau}_{ij}) n_j - (T_{pj} B_{ij})_{,h} T_{hp} \right] \delta u_i da - \int_{\partial \Omega} (T_{ip} B_{pj} n_j) (T_{ih} \delta u_h^n) da \\ &- \int_{\partial \Gamma} \llbracket B_{ij} \nu_p \rrbracket \delta u_i ds, \end{aligned} \quad (3.40)$$

or equivalently, in compact form

$$\begin{aligned} \mathcal{P}^{\text{int}} = & \int_{\Omega} \langle \text{Div}(\sigma - \tilde{\tau}) \delta u \rangle dv - \int_{\partial\Omega} \langle [(\sigma - \tilde{\tau}) \cdot n - (\nabla(B \cdot T)) : T], \delta u \rangle da \\ & - \int_{\partial\Omega} \langle (T \cdot B \cdot n), T \cdot \delta(\nabla u \cdot n) \rangle da - \int_{\partial\Gamma} \langle \llbracket B \cdot \nu \rrbracket, \delta u \rangle ds, \end{aligned} \quad (3.41)$$

where we recall once again that the tensors $\tilde{\tau}$ and B are given by eqs. (3.38), (3.39).

Considering the power of external actions to take the form (3.8) imposing $\mathcal{P}^{\text{int}} + \mathcal{P}^{\text{ext}} = 0$ and localizing, one gets the bulk equations and associated traction boundary conditions for the couple stress model by means of a direct approach

$$\text{Div}(\sigma - \tilde{\tau}) + f^{\text{ext}} = 0 \quad \text{in duality of } \delta u, \quad (3.42)$$

together with the following traction boundary conditions on the portion of the boundary $\partial\Omega \setminus \bar{\Gamma}$

$$(\sigma - \tilde{\tau}) \cdot n - [\nabla(B \cdot T)] : T = t^{\text{ext}} \quad \text{in duality of } \delta u, \quad (3.43)$$

$$T \cdot B \cdot n = T \cdot g^{\text{ext}} \quad \text{in duality of } (\nabla \delta u) \cdot n \quad (3.44)$$

and finally the following condition on the boundary of the boundary $\partial\Gamma$ where traction is assigned

$$\llbracket B \cdot \nu \rrbracket = \pi^{\text{ext}} \quad \text{in duality of } \delta u. \quad (3.45)$$

Given the identification of the tensors $\tilde{\tau}$ and B with the tensors $\tilde{\mathbf{m}}$ and \tilde{m} as specified in eqs. (3.38), (3.39), the bulk equations and traction boundary conditions (3.42)-(3.45) as derived by means of a direct approach are completely equivalent to eqs. (3.31)-(3.34).

3.5 The geometric and traction, strongly independent, boundary conditions for the indeterminate couple stress model

We have proven up to now that, independently of the method that one wants to choose to obtain the correct set of bulk equations and associated boundary conditions, passing through a full second gradient approach or a direct approach based on second order tensors instead the third order ones, one finally arrives at the following complete set of boundary conditions which can be used to complement the bulk equilibrium equation (3.42) of the couple stress model:

3.5.1 Geometric (essential or kinematical), strongly independent, boundary conditions on Γ

$$\begin{aligned} u &= u^{\text{ext}} & (3 \text{ bc}) & (3.46) \\ (\mathbb{1} - n \otimes n) \cdot (\nabla u \cdot n) &= (\mathbb{1} - n \otimes n) \cdot a^{\text{ext}} & (2 \text{ bc}) \end{aligned}$$

where $u^{\text{ext}}, a^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are prescribed functions on the subportion Γ of the boundary $\partial\Omega$, where kinematical boundary conditions are assigned.

3.5.2 Traction, strongly independent, boundary conditions on $\partial\Omega \setminus \bar{\Gamma}$

Correspondingly to the geometric boundary conditions, we may prescribe the following traction boundary conditions based on (3.43) (or equivalently (3.32))

$$\left. \begin{aligned} (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} \nabla [\text{anti}(\tilde{m} \cdot n) \cdot T] : T &= t^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{anti}(\tilde{m} \cdot n) \cdot n &= (\mathbb{1} - n \otimes n) \cdot g^{\text{ext}}, \\ \frac{1}{2} \llbracket \text{anti}(\tilde{m} \cdot n) \cdot \nu \rrbracket &= \pi^{\text{ext}}, \end{aligned} \right\} \quad \begin{array}{ll} \text{on } \partial\Omega \setminus \bar{\Gamma} & (3 \text{ bc}) \\ & (2 \text{ bc}) \\ \text{on } \partial\Gamma & (3 \text{ bc}) \end{array} \quad (3.47)$$

where $t^{\text{ext}}, g^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are prescribed functions on $\partial\Omega \setminus \bar{\Gamma}$, while π^{ext} is prescribed on $\partial\Gamma$ and leads to 3 boundary conditions on the curve $\partial\Gamma$.

It can be shown (see Appendix A.6 for the proof of the needed identities (A.31) and (A.32)) that such set of traction boundary conditions can be ulteriorly simplified in the following form

$$\left. \begin{aligned} (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} \nabla [\text{anti}(\tilde{m} \cdot n) \cdot T] : T &= t^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{anti}((\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n) \cdot n &= (\mathbb{1} - n \otimes n) \cdot g^{\text{ext}}, \\ \frac{1}{2} \llbracket \text{anti}((\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n) \cdot \nu \rrbracket &= \pi^{\text{ext}}, \end{aligned} \right\} \quad \begin{array}{ll} \text{on } \partial\Omega \setminus \bar{\Gamma} & (3 \text{ bc}) \\ & (2 \text{ bc}) \\ \text{on } \partial\Gamma & (3 \text{ bc}) \end{array} \quad (3.48)$$

where $t^{\text{ext}}, g^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are prescribed functions on $\partial\Omega \setminus \bar{\Gamma}$, while π^{ext} is prescribed on $\partial\Gamma$ and leads to 3 boundary conditions on the curve $\partial\Gamma$.

4 Assessment of the strongly independent boundary conditions for the indeterminate couple stress model in a form directly comparable to Mindlin and Tiersten's ones

Given that the bulk equations (3.42) that we obtained are the same as Mindlin and Tiersten's ones, the delicate point is now to compare our boundary conditions (3.46)-(3.47) with those provided by Mindlin and Tiersten in [59]. If a proof of the equivalence of the purely kinematical boundary conditions (3.46) with those proposed by Mindlin and Tiersten has already been provided in Lemma 2.3, the equivalence between traction boundary conditions as derived with our and Mindlin's approach is not straightforward. This is why we need here to rewrite the boundary conditions (3.48) in a suitable form.

4.1 Towards a direct comparison with Mindlin's traction boundary conditions

In order to be able to directly compare the traction boundary conditions for the indeterminate couple stress model which we obtained both passing through a second gradient theory and by means of a direct approach with those proposed by Mindlin, we need to rewrite our equations in a suitable form. In this section we show some calculations which are needed in order to reach this goal.

Proposition 4.1. *For all $\tilde{m} \in \mathbb{R}^{3 \times 3}$ and for all smooth surfaces $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid F(x_1, x_2, x_3) = 0\}$, $F : \omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of class C^2 , the following identity is satisfied:*

$$\frac{1}{2} \nabla [\text{anti}(\tilde{m} \cdot n) \cdot T] : T = \frac{1}{2} n \times \nabla [\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] + \frac{1}{2} \nabla [\text{anti}(T \cdot \tilde{m} \cdot n) \cdot T] : T. \quad (4.1)$$

Proof. The proof is included in Appendix A.7 □

Remark 4.2. *From the above proposition, it follows that the first of the boundary conditions (3.47) (or equivalently (3.32)) can be finally re-written in the form*

$$(\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times [\nabla (\langle n, (\text{sym } \tilde{m}) \cdot n \rangle)] - \frac{1}{2} \nabla [\text{anti}(T \cdot \tilde{m} \cdot n) \cdot T] : T = t^{\text{ext}}. \quad (4.2)$$

In Section 2.2 we have recalled the argument of Mindlin and Tiersten and we have remarked, see (2.17), that the term $-\frac{1}{2} \nabla [\text{anti}(T \cdot \tilde{m} \cdot n) \cdot T] : T$ is absent in their formulation since it remains somehow hidden in duality of $\text{curl}(\delta u)$ which is not manipulated further in their formulation.

4.2 Final form of the strongly independent, geometric and traction boundary conditions for the indeterminate couple stress model

Basing ourselves on the previously results obtained in Subsection 3.3, we can now establish which is the set of geometric and traction boundary conditions to be used in the indeterminate couple stress model, alternatively to those proposed by Mindlin and Tiersten. As we will better point out in the remainder of this section, the boundary conditions that we derive by our direct approach are as legitimate as those proposed by Mindlin and Tiersten. Nevertheless, if in one case one can equivalently pass from one set of imposed boundary conditions to the other one, such equivalence cannot be stated for the case of mixed boundary conditions.

4.2.1 Geometric (kinematical, essential) strongly independent boundary conditions for the indeterminate couple stress model

As for the geometric boundary conditions, we recall that one can assign on $\Gamma \subset \partial\Omega$ the following conditions

$$\begin{aligned} u &= u^{\text{ext}} & (3 \text{ bc}) \\ (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n &= (\mathbb{1} - n \otimes n) \cdot a^{\text{ext}} & (2 \text{ bc}) \end{aligned} \quad (4.3)$$

where $u^{\text{ext}}, a^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are prescribed functions. Such conditions are the geometric boundary conditions which are known to be valid in the framework of second gradient theories, with the peculiarity that here only the tangent part of the normal derivative of displacement can be assigned here.

We have already shown that the fact of assigning the tangent part of $\nabla u \cdot n$ is indeed equivalent to assigning the tangent part of $\text{curl} u$, so that such set of geometric boundary conditions can be seen to be equivalent to Mindlin and Tiersten one's according to Lemma 2.16.

4.2.2 Traction strongly independent boundary conditions for the indeterminate couple stress model

As far as the traction boundary conditions are concerned, considering the manipulated form (4.2) of equation (3.48)₁, the strongly independent boundary conditions (3.48) for the indeterminate couple stress model can be finally rewritten as

$$\left. \begin{aligned} [(\sigma - \tilde{\tau}) \cdot n - \tfrac{1}{2}n \times \nabla[(n, (\text{sym } \tilde{m}) \cdot n)] \\ - \tfrac{1}{2}\nabla[(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T] : T &= t^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n &= (\mathbb{1} - n \otimes n) g^{\text{ext}} \\ \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket &= \pi^{\text{ext}}, \end{aligned} \right\} \begin{array}{ll} \text{on } \partial\Omega \setminus \bar{\Gamma} & (3 \text{ bc}) \\ & (2 \text{ bc}) \\ \text{on } \partial\Gamma & (3 \text{ bc}), \end{array} \quad (4.4)$$

where $t^{\text{ext}}, g^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are prescribed functions on $\partial\Omega \setminus \bar{\Gamma}$, while $\pi^{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is prescribed on $\partial\Gamma$ and leads to 3 boundary conditions.

In this section we have deduced the strongly independent traction boundary conditions which are coming in a natural way from second gradient elasticity and we have compared them to those presented by Mindlin and Tiersten thus showing their apparent disagreement.

5 Are Mindlin and Tiersten's weakly independent boundary conditions equivalent to our strongly independent ones?

Up to this point, we have shown that the boundary conditions derived by Mindlin and Tiersten [59] for the indeterminate couple stress model are not directly superposable to those that we obtain by means of a standard variational approach in the spirit of second gradient theories.

Even if these sets of boundary conditions are formally not the same, they both follow from the same strain energy density. The only difference that we can point out in the two approaches is related to the process of integration by parts which is performed on the action functional based upon the considered strain energy density. Indeed, Mindlin and Tiersten's boundary conditions are only "weakly independent", while those obtained by means of our direct approach can be considered to be "strongly independent" in the sense established in Subsection 1.4.

To the sake of compactness, we use in the sequel the following notations for the internal tractions and

hypertractions respectively as obtained by Mindlin and Tiersten's and our approach

$$\begin{aligned}
\tilde{t}^{\text{int}} &:= \left(\sigma - \frac{1}{2} \text{anti}(\text{Div } \tilde{m}) \right) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle], & \text{Mindlin-Tiersten's formulation} \\
t^{\text{int}} &:= \left(\sigma - \frac{1}{2} \text{anti}(\text{Div } \tilde{m}) \right) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] & \text{our formulation} \\
&\quad - \frac{1}{2} \nabla[\text{anti}((\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), & (5.5) \\
\tilde{g}^{\text{int}} &:= (\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n, & \text{Mindlin-Tiersten's formulation} \\
g^{\text{int}} &:= (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n = \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n. & \text{our formulation}
\end{aligned}$$

In the last equality for g^{int} we have used the fact that g^{int} is indeed the dual of $T \cdot \nabla(\delta u) \cdot n$ which is a vector tangent to the boundary, which is equivalent to say that the normal part of g^{int} does not intervene in the balance equations.

To the sake of simplicity, we summarize the two sets of possible geometric and traction boundary conditions as obtained by Mindlin and Tiersten and by ourselves in the following summarizing box

Table 1. Possible sets of boundary conditions in the indeterminate couple stress model.

	Mindlin and Tiersten	Our approach
Geometric	(I) $u = \tilde{u}^{\text{ext}}$, (II) $T \cdot \text{curl } u = T \cdot \tilde{a}^{\text{ext}}$	(III) $u = u^{\text{ext}}$ (IV) $T \cdot \nabla u \cdot n = T \cdot a^{\text{ext}}$
Traction	(A) $\tilde{t}^{\text{int}} = \tilde{t}^{\text{ext}}$, (B) $\tilde{g}^{\text{int}} = \tilde{g}^{\text{ext}}$	(C) $t^{\text{int}} = t^{\text{ext}}$ (D) $g^{\text{int}} = g^{\text{ext}}$
Mixed BCs 1	(I) $u = \tilde{u}^{\text{ext}}$, (B) $\tilde{g}^{\text{int}} = \tilde{g}^{\text{ext}}$	(III) $u = u^{\text{ext}}$ (D) $g^{\text{int}} = g^{\text{ext}}$
Mixed BCs 2	(II) $T \cdot \text{curl } u = T \cdot \tilde{a}^{\text{ext}}$, (A) $\tilde{t}^{\text{int}} = \tilde{t}^{\text{ext}}$	(IV) $T \cdot \nabla u \cdot n = T \cdot a^{\text{ext}}$ (C) $t^{\text{int}} = t^{\text{ext}}$

The problem now arises to establish the equivalence between analogous sets of boundary conditions in the two approaches. Since all the presented boundary conditions arise from the same strain energy density, we would naively expect a complete equivalence between the two models. We will instead show that, if a direct equivalence can be established in some cases, this is not indeed feasible for all possible sets of boundary conditions that may be introduced in couple-stress continua. More particularly, we individuate different possible sets of boundary conditions that are allowed in couple-stress continua being compatible with the Principle of Virtual Powers as settled in Mindlin's and Tiersten's and our approach respectively:

- **Fully geometric boundary conditions.** The boundary conditions (I) and (II) (Mindlin and Tiersten) **or** (III) and (IV) (our approach) are simultaneously assigned on the same portion of the boundary.
- **Fully traction boundary conditions.** The boundary conditions (A) and (B) (Mindlin and Tiersten) **or** (C) and (D) (our approach) are simultaneously assigned on the same portion of the boundary.
- **Mixed 1: displacement/double-force boundary conditions.** The boundary conditions (I) and (B) (Mindlin and Tiersten) **or** (III) and (D) (our approach) are simultaneously assigned on the same portion of the boundary.
- **Mixed 2: force/ $D^1(u)$ boundary conditions.** The boundary conditions (II) and (A) (Mindlin and Tiersten) **or** (IV) and (C) (our approach) are simultaneously assigned on the same portion of the boundary. We recall that by $D^1(u)$ we compactly indicate the operator $T \cdot \text{curl } u$ when we consider Mindlin and Tiersten's approach **or** the operator $T \cdot \nabla u \cdot n$ when considering our approach.

We explicitly remark that, in order to be consistent with the introduced Principle of Virtual Powers, when the first sets of conditions is applied on a portion Γ of the boundary, the second ones must be assigned on the portion $\partial\Omega \setminus \Gamma$. Analogously, when assigning the third set of boundary conditions on Γ , the fourth one must be assigned on $\partial\Omega \setminus \Gamma$.

In the following subsections we carefully study the four introduced cases by establishing whether Mindlin and Tiersten's approach is equivalent with our formulation of the indeterminate couple-stress model.

5.1 Fully kinematical boundary conditions

We have already shown (see Lemma 2.3) that it is equivalent to simultaneously assign the displacement and the tangential part of its curl **or** the displacement and the tangential part of its normal derivative **on the same portion of the boundary**. This means that Mindlin and Tiersten's boundary conditions (I)+(II) are completely equivalent to our conditions (III)+(IV) in the sense that one system of equations can be directly obtained from the other and vice-versa. We would like to thank an unknown reviewer for pointing out to us this equivalency.

5.2 Fully traction boundary conditions

We consider here the case in which forces and double forces are simultaneously applied **on the same portion of the boundary**. More particularly, this means that we are simultaneously applying on the same portion of the boundary conditions (A) and (B) (Mindlin and Tiersten's) **or** conditions (C) and (D) (our approach).

We start by showing that conditions (B) and (D) are equivalent. To do so, we notice that, given two vectors v and n , one can check that, according to definitions 1.3, the following equalities hold

$$(\text{anti } v)_{ij} n_j = -\epsilon_{ijk} v_k n_j = -(n \times v)_i = (v \times n)_i.$$

This means that we can write:

$$g^{\text{int}} = \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n = ((\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n) \times n,$$

so that the boundary condition (D) can be rewritten as $((\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n) \times n = g^{\text{ext}}$. Comparing this last equation with equation (B), we can finally conclude by direct inspection that equations (B) and (D) are equivalent when setting

$$g^{\text{ext}} = \tilde{g}^{\text{ext}} \times n. \quad (5.6)$$

On the other hand, eq. (C) can be rewritten as

$$\left(\sigma - \frac{1}{2} \text{anti}(\text{Div } \tilde{m}) \right) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] = t^{\text{ext}} + \frac{1}{2} \nabla [\text{anti} (T \cdot \tilde{m} \cdot n) \cdot T] : T,$$

which, considering eq. (B) can also be rewritten as

$$\left(\sigma - \frac{1}{2} \text{anti}(\text{Div } \tilde{m}) \right) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] = t^{\text{ext}} + \underbrace{\frac{1}{2} \nabla [\text{anti} (\tilde{g}^{\text{ext}}) \cdot T] : T}_{\text{already known since only the tangential derivatives are considered}}.$$

It is easy to check that this last equation is equivalent to eq. (A) when setting

$$t^{\text{ext}} = \tilde{t}^{\text{ext}} - \frac{1}{2} \nabla [\text{anti} (\tilde{g}^{\text{ext}}) \cdot T] : T. \quad (5.7)$$

We have thus proved that, in the case of fully traction boundary conditions, given a couple of tractions $(\tilde{t}^{\text{ext}}, \tilde{g}^{\text{ext}})$ in Mindlin and Tiersten's model, one can always "a priori" find a corresponding pair of tractions $(t^{\text{ext}}, g^{\text{ext}})$ in our model such that the two sets of boundary conditions (A)+(B) and (C)+(D) are equivalent thanks to the relationships (5.6) and (5.7). The converse is clearly also true.

5.3 Mixed 1: displacement/double-force boundary conditions

We treat here the case in which we simultaneously assign the displacement and the double force on the same portion of the boundary. This is equivalent to say that one is assigning eqs. (I) and (B) in the Mindlin and Tiersten's approach **or** (III) and (D) when considering our approach. As already proven in the previous subsection, the equivalence between equations (B) and (D) can be obtained by setting the relationship (5.6) between g^{ext} and \tilde{g}^{ext} . Moreover eqs. (I) and (III) are clearly equivalent when $u^{\text{ext}} = \tilde{u}^{\text{ext}}$.

5.4 Mixed 2: force/ $D^1(u)$ boundary conditions

We have shown up to now that for the three preceeding cases of boundary conditions an *a priori* equivalence can be established between Mindlin and Tiersten's and our couple stress model. More particularly, this means that, given a boundary value problem stemming from Mindlin and Tiersten's model, we can set up (thanks to suitable identifications between tractions and double tractions in the two models) another boundary value problem which give rise to the same solution. We show here that the establishment of such an *a priori* equivalence is not possible in this last "Mixed 2" case in which forces and higher derivatives of the displacement field are simultaneously assigned on the same portion of the boundary. We start by noticing that if the displacement field is not assigned on the boundary, then Lemma 2.3) is not valid any more and, as a consequence, equations (II) and (IV) are no longer equivalent. Indeed, starting from the equations (1.21), it is possible to easily deduce that the tangent part of the normal derivative and of the curl of the displacement field are respectively given by

$$T \cdot \nabla u \cdot n = \left(\frac{\partial u_\tau}{\partial x_n}, \frac{\partial u_\nu}{\partial x_n}, 0 \right)^T \quad \text{and} \quad T \cdot \text{curl } u = \left(\frac{\partial u_n}{\partial x_\nu} - \frac{\partial u_\nu}{\partial x_n}, \frac{\partial u_\tau}{\partial x_n} - \frac{\partial u_n}{\partial x_\tau}, 0 \right)^T. \quad (5.8)$$

When, as in this case, the displacement is not assigned on the boundary, both its normal and tangential derivatives are free, so that we cannot establish an *a priori* equivalence between equations (II) and (IV). The same is true when one wants to compare equations (A) and (C). In fact, we can always recognize that eq. (C) can be rewritten as:

$$\left(\sigma - \frac{1}{2} \text{anti}(\text{Div } \tilde{m}) \right) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] = t^{\text{ext}} + \frac{1}{2} \nabla [\text{anti}(\tilde{g}^{\text{ext}}) \cdot T] : T.$$

Nevertheless, contrary to the case treated in Subsection 5.2, this last equation cannot be claimed to be equivalent to eq. (A) just by setting

$$t^{\text{ext}} = \tilde{t}^{\text{ext}} - \frac{1}{2} \nabla [\text{anti}(T \cdot \tilde{m} \cdot n) \cdot T] : T = \tilde{t}^{\text{ext}} - \frac{1}{2} \nabla [\text{anti}(\tilde{g}^{\text{int}}(u)) \cdot T] : T. \quad (5.9)$$

As a matter of fact, the term $g^{\text{int}}(u)$ appearing in equation (5.9) and defined in eq. (5.5)₃ depends on the displacement field u through the couple-stress tensor \tilde{m} defined in (3.25). This means that, actually, formula (5.9) does not allow to calculate *a priori* the force t^{ext} in our model which is equivalent to an assigned \tilde{t}^{ext} in Mindlin and Tiersten's model. Of course, one could think to assign, e.g., the boundary conditions (II) and (A) on the same boundary, solve the associated boundary value problem so finding its solution u^* and then calculate the quantities a^{ext} and t^{ext} to be assigned in our model in order to give rise to the same solution u^* . The converse operation of imposing our boundary conditions (IV) and (C) in our model and then calculate *a posteriori* the quantities \tilde{t}^{ext} and \tilde{a}^{ext} to be assigned in Mindlin and Tiersten's model to obtain the same solution, can also be envisaged. In other words, we are saying that only an *a posteriori* equivalence is eventually possible in the case that the considered force/ $D^1(u)$ boundary conditions are applied on the boundary. This fact poses, at least, some philosophical problems by giving rise to the question: *how to chose among Mindlin and Tiersten's or our mixed boundary conditions?* This indeterminacy leaves many questions open concerning the physical transparency of higher gradient theories.

6 Conclusions

The present paper gives a comprehensive analysis of the indeterminate couple stress model and of the boundary conditions arising in this theory. We have seen the indeterminate couple stress model as a special case of the full strain gradient elasticity, we have directly derived the equilibrium equations and the boundary conditions

in the same spirit as in the general strain gradient elasticity approach and we have compared our approach with that proposed by Mindlin and Tiersten [59].

As a balance for the present paper we can state that an apparent inconsistency is found between the classical (Mindlin and Tiersten's) approach and our direct approach to the indeterminate couple stress model.

Indeed, if an "a priori" equivalence can be found in most cases between the two models, we point out that this is not the case when considering "mixed" boundary conditions for which "forces" and suitable combinations of first order derivatives of displacement are simultaneously assigned on the same portion of the boundary.

It turns out that for such particular mixed boundary conditions an "a priori" equivalence cannot be established between the two models.

This fact poses serious conceptual and philosophical problems concerning the transparency of the physical meaning of the boundary conditions in higher gradient models. The question remains open whether the use of one model would be preferable to the other, at least for the quoted case of mixed boundary conditions.

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Appendix

A.1 First variation of a second gradient action functional and principle of virtual power in compact form

In this section we basically propose again the calculations concerning the first variation of a second gradient action functional by means of a compact notation instead of using Levi-Civita index notation as done instead in Section 3.1. To this purpose, let us consider the second gradient energy $W(\nabla \mathbf{u}, \nabla \nabla \mathbf{u}) = W(\nabla \mathbf{u}) + W_{\text{curv}}(\nabla \nabla \mathbf{u})$ and the associated action functional in the static case (no inertia considered here)

$$\mathcal{A} = - \int_{\Omega} W(\nabla \mathbf{u}, \nabla \nabla \mathbf{u}) dv = - \int_{\Omega} [W(\nabla \mathbf{u}) + W_{\text{curv}}(\nabla \nabla \mathbf{u})] dv.$$

The first variation of the action functional can be interpreted as the power of internal actions \mathcal{P}^{int} of the considered system and can be computed as follows

$$\mathcal{P}^{\text{int}} = \delta \mathcal{A} = - \int_{\Omega} (\langle D_{\nabla \mathbf{u}} W(\nabla \mathbf{u}), \nabla \delta \mathbf{u} \rangle + \langle D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u}), \nabla \nabla \delta \mathbf{u} \rangle) dv,$$

Integrating a first time by parts and using the divergence theorem we get

$$\begin{aligned} \delta \mathcal{A} = & \int_{\Omega} \langle \text{Div}[D_{\nabla \mathbf{u}} W(\nabla \mathbf{u})], \delta \mathbf{u} \rangle dv - \int_{\partial \Omega} \langle D_{\nabla \mathbf{u}} W(\nabla \mathbf{u}) \cdot \mathbf{n}, \delta \mathbf{u} \rangle da \\ & - \int_{\partial \Omega} \langle D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u}) \cdot \mathbf{n}, \nabla \delta \mathbf{u} \rangle da + \int_{\Omega} \text{Div}[D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u})], \nabla \delta \mathbf{u} \rangle dv. \end{aligned}$$

Integrating again by parts the last bulk term we get

$$\begin{aligned} \delta \mathcal{A} = & - \int_{\partial \Omega} \langle \{D_{\nabla \mathbf{u}} W(\nabla \mathbf{u}) - \text{Div}[D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u})]\} \cdot \mathbf{n}, \delta \mathbf{u} \rangle da \\ & + \int_{\Omega} \langle \text{Div} \{D_{\nabla \mathbf{u}} W(\nabla \mathbf{u}) - \text{Div}[D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u})]\}, \delta \mathbf{u} \rangle dv - \int_{\partial \Omega} \langle D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u}) \cdot \mathbf{n}, \nabla \delta \mathbf{u} \rangle da, \end{aligned}$$

which can also be rewritten as

$$\delta \mathcal{A} = - \int_{\partial \Omega} \langle \{\sigma - \text{Div}[\mathbf{m}]\} \cdot \mathbf{n}, \delta \mathbf{u} \rangle da + \int_{\Omega} \langle \text{Div} \{\sigma - \text{Div}[\mathbf{m}]\}, \delta \mathbf{u} \rangle dv - \int_{\partial \Omega} \langle \mathbf{m} \cdot \mathbf{n}, \nabla \delta \mathbf{u} \rangle da \quad (\text{A.1})$$

if one sets

$$\sigma = D_{\nabla \mathbf{u}} W(\nabla \mathbf{u}), \quad \mathbf{m} = D_{\nabla \nabla \mathbf{u}} W_{\text{curv}}(\nabla \nabla \mathbf{u}).$$

Using the brief digression concerning differential geometry (see Section 1.3) and recalling the properties (1.12), we can now ulteriorly manipulate the last term in eq. (A.1) as follows

$$\begin{aligned} \langle \mathbf{m} \cdot \mathbf{n}, \nabla \delta \mathbf{u} \rangle &= \langle \mathbf{m} \cdot \mathbf{n}, \nabla \delta \mathbf{u} \mathbf{1} \rangle = \langle \mathbf{m} \cdot \mathbf{n}, \nabla \delta \mathbf{u} (T + Q) \rangle = \langle (\mathbf{m} \cdot \mathbf{n}) \cdot T, \nabla \delta \mathbf{u} \rangle + \langle \mathbf{m} \cdot \mathbf{n}, (\nabla \delta \mathbf{u}) \cdot Q \rangle \\ &= \langle (\mathbf{m} \cdot \mathbf{n}) \cdot T \cdot T, \nabla \delta \mathbf{u} \rangle + \langle \mathbf{m} \cdot \mathbf{n}, (\nabla \delta \mathbf{u}) \cdot Q \rangle = \langle T, T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \nabla \delta \mathbf{u} \rangle + \langle \mathbf{m} \cdot \mathbf{n}, (\nabla \delta \mathbf{u}) \cdot (n \otimes n) \rangle \\ &= \langle T, T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \nabla \delta \mathbf{u} \rangle + \langle \{n \otimes [(\nabla \delta \mathbf{u}) \cdot n]\} \cdot \mathbf{m} \cdot \mathbf{n}, \mathbf{1} \rangle = \langle T, T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \nabla \delta \mathbf{u} \rangle + \langle n \otimes \{(\mathbf{m} \cdot \mathbf{n})^T \cdot (\nabla \delta \mathbf{u}) \cdot n\}, \mathbf{1} \rangle \\ &= \langle T, T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \nabla \delta \mathbf{u} \rangle + \langle n, (\mathbf{m} \cdot \mathbf{n})^T \cdot (\nabla \delta \mathbf{u}) \cdot n \rangle = \langle T, T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \nabla \delta \mathbf{u} \rangle + \langle (\mathbf{m} \cdot \mathbf{n}) \cdot n, (\nabla \delta \mathbf{u}) \cdot n \rangle. \end{aligned} \quad (\text{A.2})$$

We can hence recognize in the last term of this formula that the normal derivative $\left(\frac{\partial \delta u}{\partial n}\right)_i = [(\nabla \delta u) \cdot n]_i = \delta u_{i,h} n_h$ of the displacement field appears. As for the other term, it can be still manipulated, suitably integrating by parts for Γ an open subset of $\partial\Omega$ and then using the surface divergence theorem (1.17), so that we can finally write

$$\begin{aligned} \int_{\partial\Omega} \langle \mathbf{m} \cdot \mathbf{n}, \nabla \delta u \rangle da &= \int_{\partial\Omega} \langle T, \nabla [T \cdot (\mathbf{m} \cdot \mathbf{n})^T \cdot \delta u] - \nabla [T \cdot (\mathbf{m} \cdot \mathbf{n})^T] \cdot \delta u \rangle da + \int_{\partial\Omega} \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \mathbf{n}, (\nabla \delta u) \cdot \mathbf{n} \rangle da \\ &= \int_{\partial\Gamma} \llbracket \langle (\mathbf{m} \cdot \mathbf{n})^T \cdot \delta u, \nu \rangle \rrbracket ds - \int_{\partial\Omega} \langle T, \nabla [T \cdot (\mathbf{m} \cdot \mathbf{n})^T] \cdot \delta u \rangle da + \int_{\partial\Omega} \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \mathbf{n}, (\nabla \delta u) \cdot \mathbf{n} \rangle da \\ &= \int_{\partial\Gamma} \llbracket \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \nu, \delta u \rangle \rrbracket ds - \int_{\partial\Omega} \langle \nabla [(\mathbf{m} \cdot \mathbf{n}) \cdot T] : T, \delta u \rangle da + \int_{\partial\Omega} \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \mathbf{n}, (\nabla \delta u) \cdot \mathbf{n} \rangle da. \end{aligned} \quad (\text{A.3})$$

The final variation of the second gradient action functional given in (A.1), can hence be finally written as

$$\begin{aligned} \delta \mathcal{A} &= \int_{\Omega} \langle \text{Div} \{ \sigma - \text{Div}[\mathbf{m}] \}, \delta u \rangle dv - \int_{\partial\Omega} \langle \{ \sigma - \text{Div}[\mathbf{m}] \} \cdot \mathbf{n} - \nabla [(\mathbf{m} \cdot \mathbf{n}) \cdot T] : T, \delta u \rangle da \\ &\quad - \int_{\partial\Omega} \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \mathbf{n}, (\nabla \delta u) \cdot \mathbf{n} \rangle da - \int_{\partial\Gamma} \llbracket \langle (\mathbf{m} \cdot \mathbf{n}) \cdot \nu, \delta u \rangle \rrbracket ds, \end{aligned} \quad (\text{A.4})$$

which is directly comparable with eq. (3.7) obtained by means of calculations via Levi-Civita index notation. If now one recalls the principle of virtual powers according to which a given system is in equilibrium if and only if the power of internal forces is equal to the power of external forces, it is straightforward that the expression (A.4) naturally suggests which is the correct expression for the power of external forces that a second gradient continuum may sustain, namely:

$$\mathcal{P}^{\text{ext}} = \int_{\Omega} \langle f^{\text{ext}}, \delta u \rangle dv + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle t^{\text{ext}}, \delta u \rangle da + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle M^{\text{ext}}, (\nabla \delta u) \cdot \mathbf{n} \rangle da + \int_{\partial\Gamma} \langle \pi^{\text{ext}}, \delta u \rangle ds,$$

where

- f^{ext} are external bulk forces (expending power on displacement),
- t^{ext} are external surface forces (expending power on displacement),
- M^{ext} are external surface double-forces (expending power on the normal derivative of displacement) and
- π^{ext} are external line forces (expending power on displacement).

Imposing that $\mathcal{P}^{\text{int}} = -\mathcal{P}^{\text{ext}}$ and localizing, one can get the strong form of the equations of motion and associated boundary conditions for a second gradient continuum.

A.2 Some useful relationships between the third order hyperstress tensor and the second order couple stress tensor for the indeterminate couple stress model

Let us consider the third order couple stress tensor

$$\begin{aligned} \tilde{\mathbf{m}} &= \alpha_1 \nabla [\text{skew}(\nabla u)] + \frac{1}{4} (\alpha_1 - \alpha_2) [\nabla (\text{Div} u) \otimes \mathbb{1} - \text{Div}(\nabla u) \otimes \mathbb{1}] \\ &\quad + \frac{1}{4} (\alpha_1 - \alpha_2) \left[(\text{Div}(\nabla u) \otimes \mathbb{1})^{T_{12}} - (\nabla(\text{Div} u) \otimes \mathbb{1})^{T_{12}} \right], \end{aligned} \quad (\text{A.5})$$

together with the second order couple stress tensor

$$\tilde{m} = \frac{\alpha_1 + \alpha_2}{2} \nabla \text{curl} u + \frac{\alpha_1 - \alpha_2}{2} \nabla (\text{curl} u)^T = \alpha_1 \text{dev sym}(\nabla \text{curl} u) + \alpha_2 \text{skew}(\nabla \text{curl} u). \quad (\text{A.6})$$

It can be checked that, suitably deriving eq. (3.17) one gets

$$(\text{Div} \tilde{\mathbf{m}})_{ij} = \tilde{m}_{ijk,k} = \frac{\alpha_1}{2} (u_{i,jkk} - u_{j,ikk}) + \frac{1}{4} (\alpha_1 - \alpha_2) [u_{p,ipj} - u_{i,ppj} + u_{j,ppi} - u_{p,jpi}] = \frac{\alpha_1 + \alpha_2}{4} (u_{i,j} - u_{j,i})_{,kk}, \quad (\text{A.7})$$

or equivalently in compact form

$$\text{Div} \tilde{\mathbf{m}} = \frac{\alpha_1 + \alpha_2}{2} \Delta (\text{skew} \nabla u). \quad (\text{A.8})$$

On the other hand, one has that

$$\begin{aligned} \frac{1}{2} [\text{anti Div} \tilde{m}]_{ij} &= -\frac{1}{2} \epsilon_{ijp} \tilde{m}_{pm,m} = -\frac{1}{2} \epsilon_{ijp} \left(\frac{\alpha_1 + \alpha_2}{2} \epsilon_{pqk} u_{k,qmm} + \frac{\alpha_1 - \alpha_2}{2} \epsilon_{mqk} u_{k,qpm} \right) \\ &= \frac{\alpha_1 + \alpha_2}{4} (\delta_{qj} \delta_{ik} - \delta_{qj} \delta_{jk}) u_{k,qmm} \\ &\quad + \frac{\alpha_1 - \alpha_2}{4} (-\delta_{mi} \delta_{qj} \delta_{kp} + \delta_{mi} \delta_{qp} \delta_{kj} + \delta_{mj} \delta_{qi} \delta_{kp} - \delta_{mj} \delta_{qp} \delta_{ki} - \delta_{mp} \delta_{qi} \delta_{kj} + \delta_{mp} \delta_{qj} \delta_{ki}) u_{k,qpm} \\ &= \frac{\alpha_1 + \alpha_2}{4} (u_{i,jmm} - u_{j,imm}) + \frac{\alpha_1 - \alpha_2}{4} (-u_{p,jpi} + u_{j,ppi} + u_{p,ipj} - u_{i,ppj} - u_{j,ipp} + u_{i,jpp}) \\ &= \frac{\alpha_1 + \alpha_2}{4} (u_{i,j} - u_{j,i})_{mm}. \end{aligned}$$

This relationship, together with (A.7) and (A.8) implies the relationship (3.26).

As for proving the equalities (3.26)-(3.29), we start remarking that

$$\begin{aligned} (\tilde{\mathbf{m}} \cdot \mathbf{n})_{ij} &= \frac{\alpha_1}{2} (u_{i,jk} - u_{j,ik}) n_k + \frac{1}{4} (\alpha_1 - \alpha_2) [u_{p,ip} \delta_{jk} - u_{i,pp} \delta_{jk} + u_{j,pp} \delta_{ik} - u_{p,jp} \delta_{ik}] n_k \\ &= \frac{\alpha_1}{2} (u_{i,j} - u_{j,i})_{,k} n_k + \frac{1}{4} (\alpha_1 - \alpha_2) [u_{p,ip} n_j - u_{i,pp} n_j + u_{j,pp} n_i - u_{p,jp} n_i], \end{aligned}$$

which in compact form equivalently reads

$$\begin{aligned} \tilde{\mathbf{m}} \cdot \mathbf{n} &= \alpha_1 [\nabla (\text{skew } \nabla u)] \cdot \mathbf{n} + \frac{1}{4} (\alpha_1 - \alpha_2) [\nabla (\text{Div } u) \otimes \mathbf{n} - \text{Div } (\nabla u) \otimes \mathbf{n}] \\ &= -\frac{1}{4} (\alpha_1 - \alpha_2) [(\nabla (\text{Div } u) \otimes \mathbf{n}) - (\text{Div } (\nabla u) \otimes \mathbf{n})]^T \\ &= \alpha_1 [\nabla (\text{skew } \nabla u)] \cdot \mathbf{n} + \frac{(\alpha_1 - \alpha_2)}{2} \text{skew } [\nabla (\text{Div } u) \otimes \mathbf{n} - \text{Div } (\nabla u) \otimes \mathbf{n}]. \end{aligned}$$

On the other hand, using the notations introduced in (1.3), one has that

$$\begin{aligned} \left[\frac{1}{2} \text{anti}(\tilde{\mathbf{m}} \cdot \mathbf{n}) \right]_{ij} &= -\frac{1}{2} \epsilon_{ijk} \tilde{m}_{km} n_m = -\frac{1}{2} \epsilon_{ijk} \left(\frac{\alpha_1 + \alpha_2}{2} \epsilon_{kpq} u_{q,pm} + \frac{\alpha_1 - \alpha_2}{2} \epsilon_{mpq} u_{q,pk} \right) n_m, \\ &= -\frac{\alpha_1 + \alpha_2}{4} (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) u_{q,pm} n_m \\ &\quad + \frac{\alpha_1 - \alpha_2}{4} (-\delta_{mi} \delta_{jp} \delta_{kq} + \delta_{mi} \delta_{qj} \delta_{kp} + \delta_{ip} \delta_{jm} \delta_{kq} - \delta_{ip} \delta_{qj} \delta_{km} - \delta_{kp} \delta_{qi} \delta_{mj} + \delta_{mk} \delta_{pj} \delta_{qi}) u_{q,pk} n_m \\ &= \frac{\alpha_1 + \alpha_2}{4} (u_{i,j} - u_{j,i})_{,m} n_m + \frac{\alpha_1 - \alpha_2}{4} (u_{i,jm} n_m - u_{j,im} n_m - u_{q,jq} n_i + u_{j,pp} n_i + u_{q,iq} n_j - u_{i,pp} n_j) \\ &= \frac{\alpha_1}{2} (u_{i,j} - u_{j,i})_{,m} n_m + \frac{\alpha_1 - \alpha_2}{4} (u_{q,iq} n_j - u_{i,pp} n_j u_{j,pp} n_i - u_{q,jq} n_i). \end{aligned}$$

This relation, when compared to (A.9) finally allows to prove that

$$\tilde{\mathbf{m}} \cdot \mathbf{n} = \frac{1}{2} \text{anti}(\tilde{\mathbf{m}} \cdot \mathbf{n}) = \alpha_1 [\nabla (\text{skew } \nabla u)] \cdot \mathbf{n} + \frac{(\alpha_1 - \alpha_2)}{2} \text{skew } [\nabla (\text{Div } u) \otimes \mathbf{n} - \text{Div } (\nabla u) \otimes \mathbf{n}].$$

It is clear that, since also in this case $\tilde{\mathbf{m}} \cdot \mathbf{n} = \frac{1}{2} \text{anti}(\tilde{\mathbf{m}} \cdot \mathbf{n})$, then (3.26)-(3.29) are straightforwardly verified.

A.3 Some alternative calculations useful to rewrite the governing equations and boundary conditions in a form which is directly comparable to Mindlin's one

In this subsection we just report some alternative calculations to obtain the same results as before, so that they are not uncontournable to the understanding of the main results of the paper. Indeed, the result of the first part of this sections can also be re-obtained remarking that

$$\begin{aligned} \nabla[(\text{anti}[\tilde{\mathbf{m}} \cdot \mathbf{n}]) \cdot T] - \nabla[(\text{anti}[(\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) \cdot \tilde{\mathbf{m}} \cdot \mathbf{n}]) \cdot T] &= \nabla[(\text{anti}[(\mathbf{n} \otimes \mathbf{n}) \cdot \tilde{\mathbf{m}} \cdot \mathbf{n}]) \cdot T] \\ &= \nabla[(\text{anti}[\mathbf{n} \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]) \cdot T] = \nabla[\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n}) \cdot T] \end{aligned} \quad (\text{A.9})$$

and

$$\begin{aligned} \text{anti}(\mathbf{n}) (\mathbf{n} \otimes \mathbf{n}) &= (\text{anti}(\mathbf{n}) \mathbf{n}) \otimes \mathbf{n} = (\mathbf{n} \times \mathbf{n}) \otimes \mathbf{n} = 0, \\ (\mathbf{n} \otimes \mathbf{n}) \text{anti}(\mathbf{n}) &= -\mathbf{n} \otimes (\text{anti}(\mathbf{n}) \mathbf{n}) = -\mathbf{n} \otimes (\mathbf{n} \times \mathbf{n}) = 0. \end{aligned} \quad (\text{A.10})$$

and also that

$$\begin{aligned} (\nabla[\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n}) \cdot T] : T)_i &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n}) (\mathbf{1} - \mathbf{n} \otimes \mathbf{n})]_{ij,k} T_{jk} \\ &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n})]_{ij,k} T_{jk} - [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n}) (\mathbf{n} \otimes \mathbf{n})]_{ij,k} T_{jk} \\ &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle \text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ij} T_{jk} + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ij} \delta_{jk} - [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ij} n_j n_k + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ik} - [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ij} n_j n_k + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= -\{ \mathbf{n} \times \nabla[\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle] \}_i - [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n})]_{ij} (\mathbf{n} \otimes \mathbf{n})_{jk} + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= -\{ \mathbf{n} \times \nabla[\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle] \}_i - [\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle]_{,k} [\text{anti}(\mathbf{n}) (\mathbf{n} \otimes \mathbf{n})]_{ik} + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk} \\ &= -\{ \mathbf{n} \times \nabla[\langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle] \}_i + \langle \mathbf{n}, (\text{sym } \tilde{\mathbf{m}}) \cdot \mathbf{n} \rangle [\text{anti}(\mathbf{n})]_{ij,k} T_{jk}. \end{aligned} \quad (\text{A.11})$$

A.4 The missing steps in Mindlin and Tiersten's classical approach

In this section, we present once again the reasoning followed by Mindlin and Tiersten to obtain their set of bulk equations and boundary conditions trying to highlight the points in which their approach had to be further developed.

We start our analysis, remarking that the quantity $\langle \tilde{m} \cdot n, (\mathbb{1} - n \otimes n) \cdot [\text{axl}(\text{skew } \nabla \delta u)] \rangle$ still does contain contributions performing work against δu alone (even though there is a projection $\mathbb{1} - n \otimes n$ involved), which can be assigned arbitrarily and are therefore somehow related to independent variation δu . This case is not similar with the Cosserat theory in which we assume a priori that displacement u and microrotation $\bar{A} \in \mathfrak{so}(3)$ are independent kinematical degrees of freedom. On the other hand, the indeterminate couple stress model is not simply obtained as a constraint Cosserat model [82], i.e. assuming that $\bar{A} = \text{axl} \text{skew } \nabla u$. In the indeterminate couple stress model the only independent kinematical degree of freedom is u . We believe that the indeterminate couple stress model constructed as a constraint Cosserat model represents only an approximation of the indeterminate couple stress model, in the sense that the boundary conditions are not correctly and completely considered.

Indeed, using the projectors ($T = \mathbb{1} - n \otimes n$ and $Q = n \otimes n$) we obtain

$$\begin{aligned} \langle [(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \text{curl } \delta u \rangle &= \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \mathbb{1}, \nabla \delta u \rangle = \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot (T + Q), \nabla \delta u \rangle \\ &= \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot T, \nabla \delta u \rangle + \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot Q, \nabla \delta u \rangle \\ &= \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot T \cdot T, \nabla \delta u \rangle + \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \nabla \delta u \cdot Q \rangle da \\ &= \langle T, T \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]^T \cdot \nabla \delta u \rangle + \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \nabla \delta u \cdot Q \rangle \\ &= -\langle T, T \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nabla \delta u \rangle + \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \nabla \delta u \cdot Q \rangle. \end{aligned} \quad (\text{A.12})$$

On the other hand we have

$$\begin{aligned} \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \nabla \delta u \cdot Q \rangle &= \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \nabla \delta u \cdot (n \otimes n) \rangle \\ &= \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &= \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &\quad + \langle (n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &= \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &\quad - \langle n \otimes [\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &= \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &\quad - \langle n \langle [\text{anti}(\tilde{m} \cdot n) \cdot n], n \rangle, \nabla \delta u \cdot n \rangle \\ &= \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle \\ &\quad - \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, n \rangle \langle n, \nabla \delta u \cdot n \rangle \end{aligned} \quad (\text{A.13})$$

and, moreover,

$$\underbrace{\langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, n \rangle}_{\in \mathfrak{so}(3)} = 0. \quad (\text{A.14})$$

Hence

$$\langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n], \nabla \delta u \cdot Q \rangle = \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle. \quad (\text{A.15})$$

Thus, we deduce

$$\begin{aligned} \langle \tilde{m} \cdot n, \underbrace{(\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u}_{\substack{\text{not completely independent} \\ \text{second order variation}}} \rangle &= -\langle T, T \cdot \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nabla \delta u \rangle \\ &\quad + \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \underbrace{\nabla \delta u \cdot n}_{\substack{\text{completely } \delta u \text{-independent second} \\ \text{order normal variation of gradient}}} \rangle. \end{aligned} \quad (\text{A.16})$$

Since

$$\begin{aligned} \{ \underbrace{\nabla [T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot \delta u]}_{\in \mathbb{R}^3} \}_{ik} &= \{ T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot \delta u \}_{i,k} \\ &= \{ (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij} (\delta u)_j \}_{i,k} \\ &= (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} (\delta u)_j + (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij} (\delta u)_{j,k} \\ &= (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} (\delta u)_j + (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij} (\nabla \delta u)_{jk} \\ &= (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} (\delta u)_j + \{ T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot \nabla \delta u \}_{ik} \end{aligned} \quad (\text{A.17})$$

using the surface divergence theorem⁵ we obtain

$$\begin{aligned}
\int_{\partial\Omega} \langle \tilde{m} \cdot n, (\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u \rangle da &= - \int_{\partial\Omega} \langle T, \nabla [T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot \delta u] \rangle da \\
&+ \int_{\partial\Omega} T_{ik} (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} (\delta u)_j da \\
&+ \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle da \\
&\stackrel{\text{surface divergence}}{=} - \int_{\partial\Gamma} \llbracket \langle \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \delta u, \nu \rangle \rrbracket ds \\
&+ \int_{\partial\Omega} (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} T_{ik} (\delta u)_j da \\
&+ \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle da \\
&= \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle da \\
&+ \int_{\partial\Omega} (T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} T_{ik} (\delta u)_j da \\
&+ \int_{\partial\Gamma} \langle \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket, \delta u \rangle ds.
\end{aligned} \tag{A.18}$$

On the other hand, we have

$$\begin{aligned}
(T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij} &= T_{il} (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n])_{lj} \\
&= -T_{il} (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n])_{jl} = -T_{li} (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n])_{jl} \\
&= -(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n])_{jl} T_{li} = -\{(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T\}_{ji}.
\end{aligned} \tag{A.19}$$

Hence, we deduce

$$\begin{aligned}
(T \cdot (\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]))_{ij,k} T_{ik} &= -\{(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T\}_{ji,k} T_{ik} \\
&= -(\nabla[(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T])_j.
\end{aligned} \tag{A.20}$$

In view of the above computations, we deduce

$$\begin{aligned}
& - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da \\
&= - \int_{\partial\Omega} \langle \underbrace{(\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle]}_{\text{classical first boundary term of Mindlin, Gao, Yang, etc.}}, \delta u \rangle da - \int_{\partial\Omega} \langle \underbrace{\tilde{m} \cdot n, (\mathbb{1} - n \otimes n) \cdot [\text{axl}(\text{skew } \nabla \delta u)]}_{\text{classical second boundary term of Mindlin, Gao, Yang, etc.}} \rangle da \\
&\quad \dots \text{must be split further to obtain strongly independent variations...}
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
&= - \int_{\partial\Omega} \langle (\sigma - \frac{1}{2} \text{anti Div}[\tilde{m}]) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] - \frac{1}{2} \nabla[(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T] : T, \delta u \rangle da \\
&\quad - \frac{1}{2} \int_{\partial\Omega} \langle \underbrace{(\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n}_{\text{completely } \delta u\text{-independent second order normal variation of gradient}} \rangle da \\
&\quad - \frac{1}{2} \int_{\partial\Gamma} \langle \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket, \delta u \rangle ds
\end{aligned} \tag{A.22}$$

for all variations $\delta u \in C^\infty(\Omega)$. Note that $\partial(\partial\Omega \setminus \bar{\Gamma}) = \partial\Gamma$. Hence, there are indeed two terms

$$(\sigma - \frac{1}{2} \text{anti Div}[\tilde{m}]) \cdot n - \frac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] - \frac{1}{2} \nabla[(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot T] : T$$

and

$$\llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket$$

which perform work against δu , while only the term

$$(\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n$$

is related solely to the independent second order normal variation of the gradient $\nabla \delta u \cdot n$. This split of the boundary condition is not the one as obtained e.g. by Gao and Park [75] among others and seems to be entirely new in the context of the couple stress model.

⁵The surface divergence theorem in this context reads $\int_{\partial\Omega} \langle \nabla(Tv), T \rangle da = \int_{\partial\Gamma} \llbracket \langle v, \nu \rangle \rrbracket ds$, see (1.15).

A.5 The proof of Lemma 2.3

In other words, this lemma implies that, for any tangential vector field τ_i on $\Gamma \subset \partial\Omega$

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \langle \text{curl } u, \tau_i \rangle|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \langle \nabla u \cdot n, \tau_i \rangle|_{\Gamma} &= \text{known}, \end{aligned} \right. \quad (\text{A.23})$$

Since $u \in C^\infty(\Omega)$, on the one hand we obtain

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}} \quad (\text{Stokes}) \Rightarrow \text{also } \langle \text{curl } u, n \rangle|_{\Gamma} = \text{known}, \\ \langle \text{curl } u, \tau_i \rangle|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \text{curl } u|_{\Gamma} &= \text{known} \end{aligned} \right. \quad (\text{A.24})$$

$$\Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \text{skew } \nabla u|_{\Gamma} &= \text{known}. \end{aligned} \right.$$

On the other hand we deduce

$$u|_{\Gamma} = u^{\text{ext}} \quad (\Rightarrow \text{also } \nabla u \cdot \tau_i|_{\Gamma} = \text{known}) \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \langle \nabla u \cdot \tau_i, \tau_i \rangle|_{\Gamma} &= \text{known} \\ \langle \tau_i, (\nabla u)^T \cdot \tau_i \rangle|_{\Gamma} &= \text{known}, \\ \langle \nabla u \cdot \tau_i, n \rangle|_{\Gamma} &= \text{known}, \\ \langle \tau_i, (\nabla u)^T \cdot n \rangle|_{\Gamma} &= \text{known} \end{aligned} \right. \quad (\text{A.25})$$

$$\langle \nabla u \cdot n, \tau_i \rangle|_{\Gamma} = \langle b_0, \tau_i \rangle \Leftrightarrow \langle n, (\nabla u)^T \cdot \tau_i \rangle|_{\Gamma} = \langle b_0, \tau_i \rangle$$

$$\Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\nabla u)^T \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\nabla u) \cdot \tau_i|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{sym } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}. \end{aligned} \right.$$

Until now, we have obtained

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \nabla u \cdot \tau_i|_{\Gamma} &= \text{known}, \\ \text{skew } \nabla u|_{\Gamma} &= \text{known} \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \nabla u \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{skew } \nabla u) \cdot n|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ \nabla u \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{sym } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{skew } \nabla u) \cdot n|_{\Gamma} &= \text{known}, \end{aligned} \right. \quad (\text{A.26})$$

while

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\partial\Omega} &= u^{\text{ext}}, \\ \nabla u \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \\ (\text{sym } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}. \end{aligned} \right. \quad (\text{A.27})$$

The only one boundary condition which appears in the last term of (A.26) but does not appear in the last term of (A.27) is $(\text{skew } \nabla u) \cdot n|_{\Gamma} = \text{known}$. However, $(\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} = \text{known}$ implies also $(\text{skew } \nabla u) \cdot n|_{\Gamma} = \text{known}$, since

$$\left. \begin{aligned} \langle (\text{skew } \nabla u) \cdot n, n \rangle &= 0 \quad \text{always}, \\ \langle (\text{skew } \nabla u) \cdot n, \tau_i \rangle &= -\langle n, (\text{skew } \nabla u) \cdot \tau_i \rangle = \text{known}. \end{aligned} \right\} \Leftrightarrow (\text{skew } \nabla u) \cdot n = \text{known}. \quad (\text{A.28})$$

Therefore, by eliminating the redundant information we obtain

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \end{aligned} \right. \quad (\text{A.29})$$

and

$$\left. \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} &= \text{known} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} u|_{\Gamma} &= u^{\text{ext}}, \\ (\text{skew } \nabla u) \cdot \tau_i|_{\Gamma} &= \text{known}, \end{aligned} \right. \quad (\text{A.30})$$

and the proof is complete.

A.6 The proof of some identities to ulteriorly simplify the traction boundary conditions

In this appendix we prove some identities which must be used in order to show the equivalence between some traction boundary conditions. We start by checking that the following identity holds

$$\frac{1}{2}(\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n = \frac{1}{2}(\mathbb{1} - n \otimes n) \cdot \text{anti}(\tilde{m} \cdot n) \cdot n. \quad (\text{A.31})$$

In other words, we have to check if

$$(\mathbb{1} - n \otimes n) \cdot \text{anti}[(n \otimes n) \cdot \tilde{m} \cdot n] \cdot n = 0.$$

But this fact follows immediately from

$$\text{anti}[(n \otimes n) \cdot \tilde{m} \cdot n] \cdot n = \text{anti}[n \langle n, \tilde{m} \cdot n \rangle] \cdot n = \langle n, \tilde{m} \cdot n \rangle \text{anti}[n] \cdot n = \langle n, \tilde{m} \cdot n \rangle n \times n = \langle n, \tilde{m} \cdot n \rangle 0 = 0.$$

The final step in order to be able to complete our comparison to Mindlin and Tiersten's boundary conditions, is to prove that

$$\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket = \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket \quad (\text{A.32})$$

holds also true pointwise. To this aim, let us first remark that

$$\begin{aligned} [\text{anti}[(n \otimes n) \cdot \tilde{m} \cdot n]]^\pm \cdot \nu^\pm &= [\text{anti}[n \langle n, \tilde{m} \cdot n \rangle]]^\pm \cdot \nu^\pm = \langle n, [\tilde{m} \cdot n]^\pm \rangle [\text{anti}(n)]^\pm \cdot \nu^\pm \\ &= \langle n, [\tilde{m} \cdot n]^\pm \rangle (n \times \nu^\pm) = \langle n, [\tilde{m} \cdot n]^\pm \rangle \tau^\pm, \end{aligned} \quad (\text{A.33})$$

where $\tau^\pm = n \times \nu^\pm$ is tangent to curves $\partial\Gamma$, according to the orientation on $\partial\Omega \setminus \bar{\Gamma}$ and Γ , respectively.

Hence, we deduce

$$[\text{anti}[(n \otimes n) \cdot \tilde{m} \cdot n]]^+ \cdot \nu^+ + [\text{anti}[(n \otimes n) \cdot \tilde{m} \cdot n]]^- \cdot \nu^- = \langle n, [\tilde{m} \cdot n]^+ \rangle \tau^+ + \langle n, [\tilde{m} \cdot n]^- \rangle \tau^-. \quad (\text{A.34})$$

Therefore, we have

$$\begin{aligned} \llbracket [\text{anti}[\tilde{m} \cdot n] \cdot \nu] \rrbracket &= ([\text{anti}[\tilde{m} \cdot n]]^+ - [\text{anti}[\tilde{m} \cdot n]]^-) \cdot \nu \\ &= ([\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]]^+ - [\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]]^-) \cdot \nu \\ &\quad - (\langle n, [\tilde{m} \cdot n]^+ \rangle \tau^+ + \langle n, [\tilde{m} \cdot n]^- \rangle \tau^-) \\ &= \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \tilde{m} \cdot n] \cdot \nu \rrbracket - (\langle n, [\tilde{m} \cdot n]^+ \rangle \tau^+ + \langle n, [\tilde{m} \cdot n]^- \rangle \tau^-). \end{aligned} \quad (\text{A.35})$$

Using Stokes theorem and the divergence theorem, we obtain

$$\begin{aligned} &\int_{\partial\Gamma} \langle \langle n, [\tilde{m} \cdot n]^+ \rangle \tau^+, \delta u \rangle ds + \int_{\partial\Gamma} \langle \langle n, [\tilde{m} \cdot n]^- \rangle \tau^-, \delta u \rangle ds \\ &= \int_{\partial\Gamma} \langle \tau^+, \langle n, [\tilde{m} \cdot n]^+ \rangle \delta u \rangle ds + \int_{\partial\Gamma} \langle \tau^-, \langle n, [\tilde{m} \cdot n]^- \rangle \delta u \rangle ds \\ &= \int_{\partial\Omega \setminus \Gamma} \langle n, \text{curl}(\langle n, \tilde{m} \cdot n \rangle \delta u) \rangle da + \int_{\Gamma} \langle n, \text{curl}(\langle n, \tilde{m} \cdot n \rangle \delta u) \rangle da \\ &= \int_{\partial\Omega} \langle n, \text{curl}(\langle n, \tilde{m} \cdot n \rangle \delta u) \rangle da = \int_{\partial\Omega} \text{div}[\text{curl}(\langle n, \tilde{m} \cdot n \rangle \delta u)] dv = 0. \end{aligned} \quad (\text{A.36})$$

Therefore, we have

$$\int_{\partial\Gamma} \langle \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket, \delta u \rangle ds = 0, \quad (\text{A.37})$$

for all $\delta u \in C^\infty(\bar{\Omega})$. We choose

$$\delta u = \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket \quad (\text{A.38})$$

and we obtain

$$\int_{\partial\Gamma} \|\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket\|^2 ds = 0. \quad (\text{A.39})$$

Let us consider an arbitrary parametrization $\gamma : [a, b] \rightarrow \partial\Gamma$ of the curve $\partial\Gamma$. We obtain

$$\int_a^b \left(\|\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket\|(\gamma(s)) |\gamma'(s)| \right)^2 ds = 0,$$

which implies

$$\|\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket\|(\gamma(s)) = 0, \quad \forall s \in [a, b].$$

Hence

$$\|\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket - \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket\| = 0 \quad \text{on } \partial\Gamma$$

which is equivalent to

$$\llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket = \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot \nu \rrbracket \quad \text{on } \partial\Gamma. \quad (\text{A.40})$$

A.7 The proof of Proposition 4.1

First, we prove the following lemma:

Lemma A.1. *Let us consider a smooth level surface $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid F(x_1, x_2, x_3) = 0\}$, $F : \omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ of class C^2 , then it holds true:*

$$\nabla[\text{anti}(n)] : T = 0. \quad (\text{A.41})$$

Proof. Let us first remark that $\nabla[\text{anti}(n)] : T$ is well defined, since only the tangential derivative of $\text{anti}(n)$ are involved. The normal vector to the surface Σ is given by

$$n = \frac{\nabla F}{\|\nabla F\|}, \quad (n_1, n_2, n_3) = \frac{1}{\sqrt{F_{,l}F_{,l}}} (F_{,1}, F_{,2}, F_{,3}). \quad (\text{A.42})$$

Let us remark that

$$\begin{aligned} [\nabla[\text{anti}(n)] : T]_i &= [\text{anti}(n)]_{ij,k} T_{jk} = \left[\text{anti} \left(\frac{\nabla F}{\|\nabla F\|} \right) \right]_{ij,k} T_{jk} = [\|\nabla F\|^{-1} \text{anti}(\nabla F)]_{ij,k} T_{jk} \\ &= [\|\nabla F\|^{-1}]_{,k} [\text{anti}(\nabla F)]_{ij} T_{jk} + \|\nabla F\|^{-1} [\text{anti}(\nabla F)]_{ij,k} T_{jk}. \end{aligned} \quad (\text{A.43})$$

We compute

$$\begin{aligned} [\|\nabla F\|^{-1}]_{,k} &= [(F_{,l}F_{,l})^{-1/2}]_{,k} = -\frac{1}{2} [(F_{,l}F_{,l})^{-3/2}] 2 (F_{,l}F_{,lk}) = -\|\nabla F\|^{-3} F_{,l}F_{,lk}, \\ \text{anti}(n)_{ij,k} &= -[\epsilon_{ijl}F_{,l}]_{,k} = -\epsilon_{ijl}F_{,lk}, \\ T_{jk} &= \delta_{jk} - n_j n_k = \delta_{jk} - \|\nabla F\|^{-2} F_{,j}F_{,k}. \end{aligned} \quad (\text{A.44})$$

Moreover, using (A.10), we have

$$\begin{aligned} [\text{anti}(\nabla F)]_{ij} T_{jk} &= [\text{anti}(\nabla F) \cdot T]_{ik} = \|\nabla F\| [\text{anti}(n) \cdot T]_{ik} \\ &= \|\nabla F\| [\text{anti}(n) \cdot (\mathbf{1} - n \otimes n)]_{ik} = \|\nabla F\| [\text{anti}(n)]_{ik} = -\|\nabla F\| \epsilon_{ikl} n_l = -\epsilon_{ikl} F_{,l}. \end{aligned} \quad (\text{A.45})$$

Using (A.44)–(A.45) in (A.43), we obtain

$$\begin{aligned} [\nabla[\text{anti}(n)] : T]_i &= (-\|\nabla F\|^{-3} F_{,l}F_{,lk}) (-\epsilon_{iks}F_{,s}) + \|\nabla F\|^{-1} (-\epsilon_{ijl}F_{,lk}) (\delta_{jk} - \|\nabla F\|^{-2} F_{,j}F_{,k}) \\ &= \|\nabla F\|^{-3} (\epsilon_{iks}F_{,s}F_{,l}F_{,lk} + \epsilon_{ijl}F_{,lk}F_{,j}F_{,k}) - \|\nabla F\|^{-1} \epsilon_{ijl}F_{,lk}\delta_{jk} \\ &= \|\nabla F\|^{-3} (\epsilon_{ikj}F_{,j}F_{,l}F_{,lk} + \epsilon_{ijl}F_{,lk}F_{,j}F_{,k}) - \|\nabla F\|^{-1} \epsilon_{ikl}F_{,lk} \\ &= \|\nabla F\|^{-3} (\epsilon_{ilj}F_{,j}F_{,k}F_{,kl} + \epsilon_{ijl}F_{,lk}F_{,j}F_{,k}) \\ &= \|\nabla F\|^{-3} (\epsilon_{ilj} + \epsilon_{ijl}) F_{,lk}F_{,j}F_{,k} = 0, \end{aligned} \quad (\text{A.46})$$

and the proof is complete. \square

We now proceed with the proof of Proposition 4.1: We start remarking that, using the properties of the projectors T and Q introduced in (1.12) and the definition (3.39) of the tensor B one has

$$\begin{aligned} B &= \frac{1}{2} \text{anti}(\tilde{m} \cdot n) = \frac{1}{2} \text{anti}((T + Q) \cdot \tilde{m} \cdot n) = \frac{1}{2} \text{anti}(T \cdot \tilde{m} \cdot n) + \frac{1}{2} \text{anti}(Q \cdot \tilde{m} \cdot n) \\ &= \frac{1}{2} \text{anti}(T \cdot \tilde{m} \cdot n) + \frac{1}{2} \text{anti}(n \otimes n \cdot \tilde{m} \cdot n) \\ &= \frac{1}{2} \text{anti}(T \cdot \tilde{m} \cdot n) + \frac{1}{2} \text{anti}(\langle n, (\text{sym } \tilde{m}) \cdot n \rangle n) =: A + \frac{1}{2} \text{anti}(\psi \cdot n). \end{aligned}$$

Recalling the definition (1.3) of the anti-operator, the second term in the traction boundary condition (3.43) can hence be manipulated as follows:

$$[\nabla(B \cdot T) : T]_i = \left(A_{ij} T_{jp} - \frac{1}{2} \epsilon_{ijk} \psi n_k T_{jp} \right)_{,h} T_{hp} = (A_{ij} T_{jp})_{,h} T_{hp} - \frac{1}{2} (\epsilon_{ijk} \psi n_k T_{jp})_{,h} T_{hp}. \quad (\text{A.47})$$

We can now remark that

$$(\epsilon_{ijk} \psi n_k T_{jp})_{,h} T_{hp} = \epsilon_{ijk} [\psi_{,h} T_{jh} n_k + \psi T_{jh} n_{k,h} + \psi T_{jp,h} T_{hp} n_k]. \quad (\text{A.48})$$

On the other hand, recalling that $T + Q = \mathbf{1}$ and that $Q = n \otimes n$ is a symmetric tensor, we also have

$$0 = (T_{jp} + Q_{jp})_{,h} = T_{jp,h} + n_{j,h} n_p + n_{p,h} n_j \Rightarrow T_{jp,h} = -2n_{j,h} n_p. \quad (\text{A.49})$$

Using this last equality in (A.48) and the fact that $T \cdot n = 0$ we finally have

$$\begin{aligned} (\epsilon_{ijk} \psi n_k T_{jp})_{,h} T_{hp} &= \epsilon_{ijk} [\psi_{,h} T_{jh} n_k + \psi T_{jh} n_{k,h} - 2\psi n_{j,h} n_p T_{hp} n_k] \\ &= \epsilon_{ijk} \psi_{,h} T_{jh} n_k + \epsilon_{ijk} \psi T_{jh} n_{k,h} - 0, \end{aligned} \quad (\text{A.50})$$

and we can hence rewrite the term in (A.47) as

$$\begin{aligned}
[\nabla (B \cdot T) : T]_i &= \left(A_{ij} T_{jp} - \frac{1}{2} \epsilon_{ijk} \psi n_k T_{jp} \right)_{,h} T_{hp} \\
&= (A_{ij} T_{jp})_{,h} T_{hp} - \frac{1}{2} (\epsilon_{ijk} \psi_{,h} T_{jh} n_k + \epsilon_{ijk} \psi T_{jh} n_{k,h}) \\
&= (A_{ij} T_{jp})_{,h} T_{hp} + \frac{1}{2} (\epsilon_{ijk} \psi_{,h} (\delta_{jh} - T_{jh} - \delta_{jh}) n_k) - \frac{1}{2} (\epsilon_{ijk} \psi T_{jh} n_{k,h}) \\
&= (A_{ij} T_{jp})_{,h} T_{hp} + \frac{1}{2} (\epsilon_{ijk} \psi_{,h} n_j n_h n_k) - \frac{1}{2} (\epsilon_{ijk} \psi_{,j} n_k) - \frac{1}{2} (\epsilon_{ijk} \psi T_{jh} n_{k,h}) \\
&= (A_{ij} T_{jp})_{,h} T_{hp} + 0 - \frac{1}{2} (\epsilon_{ijk} \psi_{,j} n_k) - \frac{1}{2} (\epsilon_{ijk} \psi T_{jh} n_{k,h}) \\
&= (A_{ij} T_{jp})_{,h} T_{hp} - \frac{1}{2} (\epsilon_{ijk} \psi_{,j} n_k) - \frac{1}{2} (\epsilon_{ijk} \psi T_{jh} n_{k,h})
\end{aligned} \tag{A.51}$$

or equivalently in compact form (see the definition of vector product in (1.3)):

$$\begin{aligned}
\nabla (B \cdot T) : T &= \nabla (A \cdot T) : T - \frac{1}{2} (\nabla \psi) \times n - \frac{1}{2} \psi \nabla [\text{anti}(n)] : T \\
&= \nabla (A \cdot T) : T + \frac{1}{2} n \times (\nabla \psi \cdot T) - \frac{1}{2} \psi \nabla [\text{anti}(n)] : T = \\
&= \frac{1}{2} \nabla [\text{anti}(T \cdot \tilde{m} \cdot n) : T] : T + \frac{1}{2} n \times [\nabla (\langle n, (\text{sym } \tilde{m}) \cdot n \rangle)] - \frac{1}{2} \psi \nabla [\text{anti}(n)] : T.
\end{aligned} \tag{A.52}$$

Moreover, using Lemma A.1, we have

$$\nabla [\text{anti}(n)] : T = 0. \tag{A.53}$$

Therefore, the proof is complete.

A.8 Some lemmas useful to understand Mindlin and Tiersten's approach

Lemma A.2. *Let Ω be an open subset of \mathbb{R}^3 , Γ an open subsets of $\partial\Omega$ and $\tilde{t}, \tilde{g} : \partial\Omega \setminus \bar{\Gamma} \rightarrow \mathbb{R}^3$ two functions. Then, the equality*

$$\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t}, \delta u \rangle da + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u \rangle da = 0 \tag{A.54}$$

holds for all $\delta u \in C^2(\bar{\Omega})$ if and only if $\tilde{t}|_{\partial\Omega \setminus \bar{\Gamma}} = 0$ and $(\mathbb{1} - n \otimes n) \tilde{g}|_{\partial\Omega \setminus \bar{\Gamma}} = 0$.

Proof. Similar calculations as in Appendix A.4 lead to

$$\begin{aligned}
&\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t}, \delta u \rangle da + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u \rangle da \\
&= \int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t} - \nabla [(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]) \cdot T] : T, \delta u \rangle da \\
&\quad - \int_{\partial\Omega \setminus \bar{\Gamma}} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n, \nabla \delta u \cdot n \rangle da \\
&\quad - \int_{\partial\partial\Omega \setminus \bar{\Gamma}} \langle \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot \nu \rrbracket, \delta u \rangle ds
\end{aligned} \tag{A.55}$$

for all variations $\delta u \in C^2(\bar{\Omega})$. Therefore

$$\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t}, \delta u \rangle da + \int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \cdot \text{curl } \delta u \rangle da = 0 \tag{A.56}$$

implies

$$\begin{aligned}
&\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t} - \frac{1}{2} \nabla [(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]) \cdot T] : T, \delta u \rangle da \\
&\quad - \int_{\partial\Omega \setminus \bar{\Gamma}} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n, \nabla \delta u \cdot n \rangle da \\
&\quad - \int_{\partial\partial\Omega \setminus \bar{\Gamma}} \langle \llbracket \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot \nu \rrbracket, \delta u \rangle ds = 0
\end{aligned} \tag{A.57}$$

for all variations $\delta u \in C^2(\bar{\Omega})$. Since $\delta u \in C^2(\bar{\Omega})$, this fact is equivalent to

$$\begin{aligned}
&\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t} - \nabla [(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]) \cdot T] : T, \delta u \rangle da \\
&\quad - \int_{\partial\Omega \setminus \bar{\Gamma}} \underbrace{\langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n, \nabla \delta u \cdot n \rangle}_{\text{completely } \delta u\text{-independent second order normal variation of gradient}} da = 0
\end{aligned} \tag{A.58}$$

for all variations $\delta u \in C^2(\bar{\Omega})$. Having now the possibility to consider independent variations of δu and $\nabla \delta u \cdot n$ on $\partial\Omega \setminus \bar{\Gamma}$, we obtain that

$$\begin{aligned} \tilde{t} - \nabla[(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]) \cdot T] : T \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= 0, \\ (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= 0. \end{aligned} \quad (\text{A.59})$$

We also remark that

$$(\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n = \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n. \quad (\text{A.60})$$

Hence, it remains to prove that from

$$\begin{aligned} \tilde{t} - \nabla[\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot T] : T \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= 0, \\ \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= 0 \end{aligned} \quad (\text{A.61})$$

it follows that $\tilde{t} \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$ and $(\mathbb{1} - n \otimes n) \cdot \tilde{g} \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$.

In the suitable local coordinate system (τ, ν, n) at the boundary we obtain

$$\begin{aligned} n \otimes n &= \text{diag}(0, 0, 1), \quad (\mathbb{1} - n \otimes n) = \text{diag}(1, 1, 0), \quad \tilde{g} = \tilde{g}_\tau \tau + \tilde{g}_\nu \nu + \tilde{g}_n n, \\ (\mathbb{1} - n \otimes n) \tilde{g} &= \tilde{g}_\tau \tau + \tilde{g}_\nu \nu, \quad [(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \times n = \tilde{g}_\tau \nu - \tilde{g}_\nu \tau. \end{aligned} \quad (\text{A.62})$$

Hence

$$\begin{aligned} \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot n \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= (\tilde{g}_\tau \nu - \tilde{g}_\nu \tau) \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0 \\ \Leftrightarrow \tilde{g}_\tau \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= 0, \quad \tilde{g}_\nu \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0 \quad \Leftrightarrow \quad (\mathbb{1} - n \otimes n) \cdot \tilde{g} \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0. \end{aligned} \quad (\text{A.63})$$

Moreover, we deduce

$$\begin{aligned} \{\nabla[\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}] \cdot T] : T\}_i \Big|_{\partial\Omega \setminus \bar{\Gamma}} &= [\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]_{ij,k} T_{jk}] \Big|_{\partial\Omega \setminus \bar{\Gamma}} = [\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]_{ij,k} \tau_k \tau_j] \Big|_{\partial\Omega \setminus \bar{\Gamma}} \\ &= \langle \nabla\{\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]_{ij}\}, \tau \rangle \tau_j \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0, \end{aligned} \quad (\text{A.64})$$

since $[\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{g}]_{ij}] \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$, for all $i, j = 1, 2, 3$. Therefore, it follows that also $\tilde{t} \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$. \square

Lemma A.3. *Let Ω be an open subset of \mathbb{R}^3 , Γ an open subsets of $\partial\Omega$ and $\tilde{t} : \partial\Omega \setminus \bar{\Gamma} \rightarrow \mathbb{R}^3$, $\tilde{g} : \partial\Omega \setminus \bar{\Gamma} \rightarrow \mathbb{R}^3$ two functions. Then, the equality*

$$\int_{\partial\Omega \setminus \bar{\Gamma}} \langle \tilde{t}, \delta u \rangle da + \int_{\Gamma} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \cdot \text{curl} \delta u \rangle da = 0 \quad (\text{A.65})$$

holds for all $\delta u \in C^2(\bar{\Omega})$, $\delta u \Big|_{\Gamma} = 0$ and $(\mathbb{1} - n \otimes n) \cdot \text{curl} \delta u \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$ if and only if $\tilde{t} \Big|_{\partial\Omega \setminus \bar{\Gamma}} = 0$ and $(\mathbb{1} - n \otimes n) \tilde{g} \Big|_{\Gamma} = 0$.

Proof. The proof is similar with the proof of Lemma A.2. \square

A.9 Concluding diagrams

Standard boundary conditions in the indeterminate couple stress model [59]

Geometric (essential) boundary conditions (3+2) **[only weakly independent]**

$$u|_{\Gamma} = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \tilde{a}^{\text{ext}} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \tilde{b}^{\text{ext}}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} ((\sigma - \tilde{\tau}) \cdot n - \tfrac{1}{2} n \times \nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle])|_{\partial\Omega \setminus \bar{\Gamma}} &= t^{\text{ext}}, & \tilde{\tau} = \text{Div } \tilde{\mathfrak{m}} = \tfrac{1}{2} \text{anti}(\text{Div } \tilde{m}) \in \mathfrak{so}(3) & \quad 3 \text{ bc} \\ (\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n|_{\partial\Omega \setminus \bar{\Gamma}} &= (\mathbb{1} - n \otimes n) \cdot m^{\text{ext}} & & \quad 2 \text{ bc} \end{aligned}$$

Boundary virtual work

$$\begin{aligned} - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da &= 0 \quad \Leftrightarrow \\ - \int_{\partial\Omega} \left\langle \left((\sigma - \tilde{\tau}) \cdot n - \tfrac{1}{2} n \times \underbrace{\nabla[\langle n, (\text{sym } \tilde{m}) \cdot n \rangle]}_{\text{normal curvature}} \right), \delta u \right\rangle da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \left\{ (\mathbb{1} - n \otimes n) \cdot [\text{axl}(\text{skew } \nabla \delta u)] \right\} \rangle da &= 0 \end{aligned}$$



Standard boundary conditions in the indeterminate couple stress model, index-format

Geometric (essential) boundary conditions (3+2) **[only weakly independent]**

$$\begin{aligned} u_i|_{\Gamma} = u_{,i}^{\text{ext}} \in \mathbb{R}^3, \quad (\epsilon_{ikl} u_{l,k} - \epsilon_{jkl} u_{l,k} n_j n_i)|_{\Gamma} &= \epsilon_{ikl} u_{l,k}^{\text{ext}} - \epsilon_{jkl} u_{l,k}^{\text{ext}} n_j n_i, \\ \text{or} \quad (u_{i,k} n_k - u_{j,k} n_k n_j n_i)|_{\Gamma} &= u_{i,k}^{\text{ext}} n_k - u_{j,k}^{\text{ext}} n_k n_j n_i \end{aligned}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j - \tfrac{1}{2} \epsilon_{ikl} n_k (\tilde{m}_{ij} n_i n_j)_{,l})|_{\partial\Omega \setminus \bar{\Gamma}} &= t_i^{\text{ext}}, & \tilde{\tau}_{ij} = \tfrac{1}{2} \epsilon_{jik} \tilde{m}_{kl,l} \in \mathfrak{so}(3) & \quad 3 \text{ bc} \\ (\tilde{m}_{ik} n_k - \tilde{m}_{jk} n_k n_j n_i)|_{\partial\Omega \setminus \bar{\Gamma}} &= m_i^{\text{ext}} - m_j^{\text{ext}} n_j n_i & & \quad 2 \text{ bc} \end{aligned}$$

Boundary virtual work

$$\begin{aligned} - \int_{\partial\Omega} ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j) \delta u_i da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da &= 0 \quad \Leftrightarrow \\ - \int_{\partial\Omega} ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j - \tfrac{1}{2} \epsilon_{ikl} n_k (\tilde{m}_{ij} n_i n_j)_{,l}) \delta u_i da - \tfrac{1}{2} \int_{\partial\Omega} (\tilde{m}_{ik} n_k - \tilde{m}_{jk} n_k n_j n_i) (\epsilon_{ikl} \delta u_{l,k} - \epsilon_{jkl} \delta u_{l,k} n_j n_i) da &= 0 \end{aligned}$$

Figure 2: The standard boundary conditions in the indeterminate couple stress model which have been employed hitherto by all authors to our knowledge. The virtual displacement is denoted by $\delta u \in C^\infty(\bar{\Omega})$. The number of traction boundary conditions is correct, but the split into independent variations at the boundary is not taken to its logical end.

Boundary conditions in the indeterminate couple stress model in terms of gradient elasticity and third order moment tensors

Geometric (essential) boundary conditions (3+2) **[strongly independent]**

$$u|_{\Gamma} = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \tilde{a}^{\text{ext}} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \tilde{b}^{\text{ext}}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} ((\sigma - \text{Div } \tilde{\mathbf{m}}) \cdot n - \nabla[(\tilde{\mathbf{m}} \cdot n) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n))|_{\partial\Omega \setminus \bar{\Gamma}} &= t^{\text{ext}}, & \tilde{\mathbf{m}} &= D_{\nabla \nabla u}[\tilde{W}_{\text{curv}}(\nabla[\text{axl}(\text{skew } \nabla u))]] & 3 \text{ bc} \\ (\mathbb{1} - n \otimes n) \cdot [\tilde{\mathbf{m}} \cdot n] \cdot n|_{\partial\Omega \setminus \bar{\Gamma}} &= (\mathbb{1} - n \otimes n) \cdot m^{\text{ext}} & & & 2 \text{ bc} \\ \llbracket (\tilde{\mathbf{m}} \cdot n) \cdot \nu \rrbracket|_{\partial\Gamma} &= \pi^{\text{ext}}, & & \text{“edge line force” on } \partial\Gamma & 3 \text{ bc} \end{aligned}$$

Boundary virtual work

$$\begin{aligned} - \int_{\partial\Omega} \langle (\sigma - \text{Div } \tilde{\mathbf{m}}) \cdot n, \delta u \rangle da - \int_{\partial\Omega} \langle \tilde{\mathbf{m}} \cdot n, \nabla \delta u \rangle da &= 0 \quad \Leftrightarrow \\ - \int_{\partial\Omega} \langle (\sigma - \text{Div } \tilde{\mathbf{m}}) \cdot n - \nabla[(\tilde{\mathbf{m}} \cdot n) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle da - \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \cdot [\tilde{\mathbf{m}} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle da \\ - \int_{\partial\Gamma} \langle \llbracket (\tilde{\mathbf{m}} \cdot n) \cdot \nu \rrbracket, \delta u \rangle ds &= 0 \end{aligned}$$

\Updownarrow equivalent

Boundary conditions (3+2) in the indeterminate couple stress model in terms of gradient elasticity, third order moment tensors, and written in indices

Geometric (essential) boundary conditions (3+2) **[strongly independent]**

$$\begin{aligned} u_i|_{\Gamma} &= u_i^{\text{ext}}, & (u_{i,k} n_k - u_{j,k} n_j n_i)|_{\Gamma} &= u_{i,k}^{\text{ext}} n_k - u_{j,k}^{\text{ext}} n_j n_i, \\ \text{or } (\epsilon_{ikl} u_{l,k} - \epsilon_{jkl} u_{l,k} n_j n_i)|_{\Gamma} &= (\epsilon_{ikl} u_{l,k}^{\text{ext}} - \epsilon_{jkl} u_{l,k}^{\text{ext}} n_j n_i), \end{aligned}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} \left[(\sigma_{ij} - \tilde{m}_{ijk,k}) n_j - (\tilde{m}_{ipk} n_k - \tilde{m}_{ijk} n_k n_j n_p)_{,h} (\delta_{ph} - n_p n_h) \right]|_{\partial\Omega \setminus \bar{\Gamma}} &= t_i^{\text{ext}}, & \tilde{m}_{ijk} &= D_{u_{,ijk}} \tilde{W}_{\text{curv}} \left((\epsilon_{ijk} u_{k,j})_{,m} \right) & 3 \text{ bc} \\ (\tilde{m}_{ijp} n_j - \tilde{m}_{pjk} n_k n_j n_i) n_p|_{\partial\Omega \setminus \bar{\Gamma}} &= m_i^{\text{ext}} - m_p^{\text{ext}} n_p n_i & & & 2 \text{ bc} \\ \llbracket \tilde{m}_{pjk} n_k \nu_j \rrbracket|_{\partial\Gamma} &= \pi_p^{\text{ext}}, & & \text{“edge line force” on } \partial\Gamma & 3 \text{ bc} \end{aligned}$$

Figure 3: The standard strongly independent boundary conditions in the indeterminate couple stress model in terms of a third order couple stress tensor coming from full gradient elasticity. The virtual displacement is denoted by $\delta u \in C^\infty(\bar{\Omega})$. The summation convention was used in index notations.

Strongly independent boundary conditions in the indeterminate couple stress model

Geometric (essential) boundary conditions (3+2)

$$u|_{\Gamma} = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot a^{\text{ext}} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot b^{\text{ext}}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} & \left((\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \nabla [\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] \right. \\ & \quad \left. - \frac{1}{2} \nabla [(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n) \right) \Big|_{\partial\Omega \setminus \bar{\Gamma}} = t^{\text{ext}}, \quad 3 \text{ bc} \\ & \quad (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n \Big|_{\partial\Omega \setminus \bar{\Gamma}} = (\mathbb{1} - n \otimes n) \cdot m^{\text{ext}} \quad 2 \text{ bc} \\ & \quad \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket \Big|_{\partial\Gamma} = \pi^{\text{ext}}, \quad \text{“edge line force” on } \partial\Gamma \quad 3 \text{ bc} \end{aligned}$$

Boundary virtual work

$$\begin{aligned} & - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da = 0 \quad \Leftrightarrow \\ & - \int_{\partial\Omega} \langle \left\{ (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \underbrace{\nabla [\langle n, (\text{sym } \tilde{m}) \cdot n \rangle]}_{\text{normal curvature}} \right\}, \delta u \rangle da + \int_{\partial\Omega} \langle \tilde{m} \cdot n, \left\{ (\mathbb{1} - n \otimes n) \cdot [\text{axl}(\text{skew } \nabla \delta u)] \right\} \rangle da = 0 \\ & - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} n \times \nabla [\langle n, (\text{sym } \tilde{m}) \cdot n \rangle] - \frac{1}{2} \nabla [(\text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n]) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle da \\ & \quad - \frac{1}{2} \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \cdot \text{anti}[(\mathbb{1} - n \otimes n) \cdot \tilde{m} \cdot n] \cdot n, \nabla \delta u \cdot n \rangle da - \frac{1}{2} \int_{\partial\Gamma} \langle \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket, \delta u \rangle ds = 0 \end{aligned}$$

\Updownarrow equivalent

Equivalent strongly independent boundary conditions in the indeterminate couple stress model

Geometric (essential) boundary conditions (3+2)

$$u|_{\Gamma} = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbb{1} - n \otimes n) \cdot \nabla u \cdot n|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot a^{\text{ext}} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbb{1} - n \otimes n) \cdot \text{curl } u|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot b^{\text{ext}}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} & \left((\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} \nabla [(\text{anti}(\tilde{m} \cdot n)) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n) \right) \Big|_{\partial\Omega \setminus \bar{\Gamma}} = t^{\text{ext}}, \quad 3 \text{ bc} \\ & \quad (\mathbb{1} - n \otimes n) \cdot \text{anti}[\tilde{m} \cdot n] \cdot n \Big|_{\partial\Omega \setminus \bar{\Gamma}} = (\mathbb{1} - n \otimes n) \cdot m^{\text{ext}} \quad 2 \text{ bc} \\ & \quad \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket \Big|_{\partial\Gamma} = \pi^{\text{ext}}, \quad \text{“edge line force” on } \partial\Gamma \quad 3 \text{ bc} \end{aligned}$$

Boundary virtual work

$$\begin{aligned} & - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n, \delta u \rangle da - \int_{\partial\Omega} \langle \tilde{m} \cdot n, \text{axl}(\text{skew } \nabla \delta u) \rangle da = 0 \quad \Leftrightarrow \\ & - \int_{\partial\Omega} \langle (\sigma - \tilde{\tau}) \cdot n - \frac{1}{2} \nabla [(\text{anti}(\tilde{m} \cdot n)) \cdot (\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle da \\ & \quad - \frac{1}{2} \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \text{anti}(\tilde{m} \cdot n) \cdot n, \nabla \delta u \cdot n \rangle da - \frac{1}{2} \int_{\partial\Gamma} \langle \llbracket \text{anti}[\tilde{m} \cdot n] \cdot \nu \rrbracket, \delta u \rangle ds = 0 \end{aligned}$$

\Updownarrow equivalent

Alternative equivalent strongly independent boundary conditions, index-format

Geometric (essential) boundary conditions (3+2)

$$\begin{aligned} u_i|_{\Gamma} &= u_i^{\text{ext}} \in \mathbb{R}^3, \quad (\epsilon_{ikl} u_{l,k} - \epsilon_{jkl} u_{l,k} n_j n_i) \Big|_{\Gamma} = a_i^{\text{ext}} \\ &\quad \text{or} \quad (u_{i,k} n_k - u_{j,k} n_k n_j n_i) \Big|_{\Gamma} = b_i^{\text{ext}} \end{aligned}$$

Mechanical (traction) boundary conditions (3+2)

$$\begin{aligned} & ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j + \frac{1}{2} (\epsilon_{ihk} \tilde{m}_{ks} n_s - \epsilon_{ijk} \tilde{m}_{ks} n_s n_j n_h)_{,p} (\delta_{hp} - n_h n_p)) \Big|_{\partial\Omega \setminus \bar{\Gamma}} = t_i^{\text{ext}}, \quad 3 \text{ bc} \\ & \quad (\epsilon_{ipk} \tilde{m}_{ks} n_s - \epsilon_{jpk} \tilde{m}_{ks} n_s n_j n_i) n_p \Big|_{\partial\Omega \setminus \bar{\Gamma}} = m_i^{\text{ext}} - m_p^{\text{ext}} n_p n_i, \quad 2 \text{ bc} \\ & \quad \llbracket \epsilon_{ipk} \tilde{m}_{ks} n_s \nu_p \rrbracket \Big|_{\partial\Gamma} = \pi_i^{\text{ext}}, \quad \text{“edge line force” on } \partial\Gamma \quad 3 \text{ bc} \end{aligned}$$

Figure 4: The possible boundary conditions in the indeterminate couple stress model. The equivalence of the geometric boundary condition is clear. The virtual displacement is denoted by $\delta u \in C^\infty(\bar{\Omega})$.